Let $P = (2, 1, 0)$, $Q = (1, 1, 1)$ and $R = (0, -1, 3)$ be three points in $\mathbb{R}^3$.

(1) Find the cosine of the angle between $\overrightarrow{PO}$ and $\overrightarrow{FR}$.

$\overrightarrow{PQ} = \langle -1, 0, 1 \rangle$, $\overrightarrow{PR} = \langle -2, -2, 3 \rangle$

$|\overrightarrow{PQ}| = \sqrt{2}$, $|\overrightarrow{PR}| = \sqrt{17}$

$\overrightarrow{PQ} \cdot \overrightarrow{PR} = 5$

$\cos \angle QPR = \frac{5}{\sqrt{2} \cdot \sqrt{17}} = \frac{5}{\sqrt{34}} = \frac{5\sqrt{34}}{34}$

YOUR ANSWER: \[
\cos \angle QPR = \frac{5}{\sqrt{34}}
\]

(2) Find the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$.

$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, 1, 2 \rangle$

YOUR ANSWER: \[
\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, 1, 2 \rangle
\]
(3) Find the area of the triangle with vertices \( P, Q, R \).

\[
\text{Area} = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| \\
= \frac{3}{2}
\]

\[
\text{Area}(\triangle PQR) = \frac{3}{2}
\]

YOUR ANSWER:

(4) Determine the equation of the plane \( \mathcal{P} \) through the points \( P, Q, R \) in the form \( Ax + By + Cz + D = 0 \).

\[
2(x-2) + 1(y-4) + 2(z-0) = 0 \\
(\text{or simply } Q, R)
\]

\[
2x + y + 2z - 5 = 0
\]

The equation of the plane through \( P, Q, R \) is

YOUR ANSWER:

\[
2x + y + 2z - 5 = 0
\]
(5) Find the distance from the origin to this plane $\mathcal{P}$ through the points $P, Q, R$.

$$d = \frac{|2 \cdot 0 + 1 \cdot 0 + 2 \cdot 0 - 5|}{\sqrt{2^2 + 1^2 + 2^2}}$$

$$= \frac{5}{3}$$

The distance from the origin to this plane is $\frac{5}{3}$

YOUR ANSWER: $\frac{5}{3}$

(6) For which value of $c$ is the line $\mathbf{r}(t) = (1 + ct, -3t, t)$ parallel to the plane $\mathcal{P}$ in (4)?

$$0 = \langle c, -3, 1 \rangle \cdot \langle 2, 1, 2 \rangle = 2c - 3 + 2 = 2c - 1$$

$$\Rightarrow c = \frac{1}{2}$$

YOUR ANSWER: $\frac{1}{2}$
A particle is traveling along the curve \( \mathbf{r}(t) = (\sin(\pi t), t^2 - t, \cos(\pi t)) \), where \( t \) is the time variable.

(1) Compute the velocity \( \mathbf{v}(t) \) and the speed \( s(t) \) of the particle at an arbitrary time \( t \).

\[
\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \left( \pi \cos(\pi t), 2t - 1, -\pi \sin(\pi t) \right)
\]

\[
s(t) = |\mathbf{v}(t)| = \sqrt{(2t-1)^2 + \pi^2}
\]

YOUR ANSWER:

\[
\mathbf{v}(t) = \left( \pi \cos(\pi t), 2t - 1, -\pi \sin(\pi t) \right)
\]

\[
s(t) = \sqrt{(2t-1)^2 + \pi^2}
\]

(2) At which point \( P \) will the particle meet the surface \( x^2 + 4y + z^2 = 0 \)?

\[
0 = (\sin(\pi t))^2 + 4(t^2 - t) + (\cos(\pi t))^2
\]

\[
= 4t^2 - 4t + 1
\]

\[
= (2t-1)^2
\]

\( \Rightarrow \ t = \frac{1}{2} \)

\( \Rightarrow \ P = (1, -\frac{1}{4}, 0) \)

The particle will meet the given surface at

\[
(1, -\frac{1}{4}, 0)
\]

YOUR ANSWER:
(3) Find the equation of the tangent line $L$ of the curve $\vec{r} = \vec{r}(t)$ at $P$.

\[ \vec{r}''(\frac{1}{2}) = \vec{v}(\frac{1}{2}) = \langle 0, 0, -\pi \rangle \]

\[ \Rightarrow \begin{cases} x = 1 \\ y = -\frac{1}{4} \\ z = -\pi t \end{cases} \]

The equation for the tangent line $L$ is

\[ \begin{cases} x = 1 \\ y = -\frac{1}{4} \\ z = -\pi t \end{cases} \]

YOUR ANSWER: 

(4) Find the equation of the tangent plane $P$ of the surface $x^2 + 4y + z^2 = 0$ at $P$.

\[ F(x, y, z) = x^2 + 4y + z^2 \]

\[ \nabla F = \langle 2x, 4, 2z \rangle \]

\[ \Rightarrow \nabla F(P) = \langle 2, 4, 0 \rangle \]

Plane: \[ 2(x-1) + 4(y+\frac{1}{4}) + 0(z-0) = 0 \]

\[ \Rightarrow 2x + 4y - 1 = 0 \]

The equation for the tangent plane $P$ is

\[ 2x + 4y - 1 = 0 \]

YOUR ANSWER: 

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3 (15 pts.) The diagram below gives the level sets of a function \( h = h(x, y) \). Please answer the following questions for this function.

(1) What is the value of \( h \) at the point \( A \)?

YOUR ANSWER:

At the point \( A, h(x, y) = 1 \)

(2) At which point of the points \( B \) and \( C \) is the value of \( \frac{\partial h}{\partial y} \) greater?

YOUR ANSWER:

The value of \( \frac{\partial h}{\partial y} \) is greater at \( C \)

(3) On the picture, draw a vector representing the direction of \( \nabla h \) at the point \( B \).

(4) In the given region, mark a letter \( P \) at all local maximum point(s) of the function \( h(x, y) \), mark a letter \( Q \) at all local minimum point(s) of \( h(x, y) \), and mark a letter \( R \) at all saddle point(s) of \( h(x, y) \).
4 (35 pts.) Consider the function \( f(x, y) = x^3 - 3x + y^2 \).

(1) Find the tangent plane equation for the surface \( z = f(x, y) \) at the point \((1, 1, -1)\).

\[
\frac{\partial f}{\partial x} = 3x^2 - 3 \Rightarrow \frac{\partial f}{\partial x}(1, 1) = 0
\]

\[
\frac{\partial f}{\partial y} = 2y \Rightarrow \frac{\partial f}{\partial y}(1, 1) = 2
\]

\[
z = -1 + 0(x-1) + 2(y-1) = 2y - 3
\]

The tangent plane equation is

YOUR ANSWER: \(z = 2y - 3\)

(2) Use the linear approximation to estimate \( f(1.01, 0.99) \).

\[
f(1.01, 0.99) \approx 2 \cdot 0.99 - 3 = -1.02
\]

YOUR ANSWER: \(-1.02\)
(3) Find the directional derivative of $f$ at $(1, 1)$ in the unit direction $\vec{u}$ of the vector $(3, 4)$.

$|\langle 3, 4 \rangle| = 5$

$\vec{u} = \frac{\langle 3, 4 \rangle}{5} = \langle \frac{3}{5}, \frac{4}{5} \rangle.$

$D_{\vec{u}} f(1, 1) = \nabla f(1, 1) \cdot \vec{u}$

$= \langle 0, 2 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{8}{5}$

$D_{\vec{u}} f(1, 1) = \frac{8}{5}$

YOUR ANSWER:

(4) Find the rate of change of $f$, $\frac{df}{dt}$, along the parametric curve $x(t) = 2t - 1, \quad y(t) = 3 - 2t^2$

at $t = 1$.

$\frac{df}{dt} (1) = \frac{\partial f}{\partial x}(1, 1) \frac{dx}{dt}(1) + \frac{\partial f}{\partial y}(1, 1) \frac{dy}{dt}(1)$

$= 0 \cdot 2 + 2 \cdot (-4)$

$= -8$

$\frac{df}{dt}(1) = -8$

YOUR ANSWER:
(5) Find all the critical point(s) of \( f \), and classify them as local maximum points, local minimum points, or saddle points.

\[
\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad 3x^2 - 3 = 0 \quad \Rightarrow \quad x = \pm 1
\]

\[
\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad 2y = 0 \quad \Rightarrow \quad y = 0
\]

\[\Rightarrow \text{critical points}: \ (1, \ 0), \ (-1, \ 0).\]

\[
\begin{align*}
\Delta f_{xx} &= 6, \\
\Delta f_{xy} &= 0, \\
\Delta f_{yy} &= 2.
\end{align*}
\]

At \((1, 0)\), \( D = 6 \cdot 2 - 0 = 12 > 0 \quad \Rightarrow \quad \text{local minimum} \quad \checkmark \)

At \((-1, 0)\), \( D = -6 \cdot 2 - 0 = -12 < 0 \quad \Rightarrow \quad \text{saddle} \quad \checkmark \)

\[
\text{Local maximum point(s):} \\
\text{Local minimum point(s):} \ (1, \ 0) \\
\text{Saddle point(s):} \ (-1, \ 0) \\
\]

YOUR ANSWER:
(6) Using the method of Lagrange multipliers, set up \textbf{but do not solve}
the system of equations that determine the point on the surface \( z = f(x, y) \) which is closest to the point \((1, 1, 1)\).

Want to minimize \( \phi(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2 \)
subject to condition
\[
\begin{align*}
x^3 - 3x + y^2 - z &= 0.
\end{align*}
\]

Equations:
\[
\begin{aligned}
2(x-1) &= \lambda (3x^2 - 3) \\
2(y-1) &= \lambda \cdot 2y \\
2(z-1) &= \lambda (4) = -\lambda \\
x^3 - 3x + y^2 - z &= 0
\end{aligned}
\]

The system of equations to be solved is
\[
\begin{aligned}
2x - 2 &= \lambda (3x^2 - 3) \\
2y - 2 &= \lambda \cdot 2y \\
2z - 2 &= -\lambda \\
x^3 - 3x + y^2 - z &= 0
\end{aligned}
\]