1. Consider a cube with sides of length 2; see the picture. Let $A$ and $B$ be the vertices indicated in the picture. Let $Q$ be the point which is in the middle of an edge of the box (equal distance from the two end points of the edge) as indicated in the picture.

(a) (6 points) What is the angle between the line segment $QA$ and the line segment $QB$? (You can leave it as $\cos^{-1}$ of a real number.)

$$A(2,0,2), \quad B(2,2,0), \quad Q(0,2,1)$$

$$\overrightarrow{QA} = \langle -2, 2, -1 \rangle, \quad \overrightarrow{QB} = \langle -2, 0, 1 \rangle$$

$$\cos \theta = \frac{\overrightarrow{QA} \cdot \overrightarrow{QB}}{|\overrightarrow{QA}| |\overrightarrow{QB}|} = \frac{3}{\sqrt{16} \sqrt{5}} = \frac{3}{4 \sqrt{5}}$$

The angle is $\cos^{-1} \left( \frac{3}{4 \sqrt{5}} \right)$

(b) (4 points) What is the area of the triangle formed by $A$, $B$ and $Q$?

$$\frac{1}{2} |\overrightarrow{QA} \times \overrightarrow{QB}|$$

$$\overrightarrow{QA} \times \overrightarrow{QB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -1 \\ -2 & 0 & 1 \end{vmatrix}$$

$$= \langle 2, 4, 4 \rangle$$

The area is 3
2. (10 points) Consider the surface given by the equation \( x^2 + y^2 + z^2 = 6 \) and another surface given by the equation \( z = x^2 + y^2 \). The intersection of these two surfaces forms a curve. Find the length of this curve.

\[ z^2 + z = 6 \]
\[ z^2 + z - 6 = 0 \]
\[ (z+3)(z-2) = 0 \]
\[ \Rightarrow z = 2 \]

The circle is \( x^2 + y^2 = 2 \), \( z = 2 \).

Radius = \( \sqrt{2} \)

The length is \( 2 \pi \sqrt{2} \).
3. (10 points) Two objects travel through space along two different curves. The trajectory of object $A$ is given by the function $\vec{r}_1(t) = (t^2, 7t - 12, t^3)$, $t \geq 0$, and the trajectory of object $B$ is given by the function $\vec{r}_2(t) = (4t - 3, t^2, 5t - 6)$, $t \geq 0$.

(a) Do the objects collide? (Show your work in detail.)

$$\begin{align*}
\begin{cases}
    t^2 = 4t - 3 \\
    7t - 12 = t^2 \\
    t^2 = 5t - 6
\end{cases}
\end{align*}$$

\[7t - 12 = 4t - 3 \implies t = 3\]

Your answer: \text{Yes, } a^+ t = 3

(b) If your answer to (a) is yes, determine which object was traveling at a faster speed at the collision. On the other hand, if your answer to (a) is no, determine whether the curves represented by the trajectories intersect or not.

$$\vec{v}_1(t) = <2t, 7, 2t> \quad , \quad \vec{v}_1(3) = <6, 7, 6>$$

$$|\vec{v}_1(3)| = \sqrt{121}$$

$$\vec{v}_2(t) = <4, 2t, 5> \quad , \quad \vec{v}_2(3) = <4, 6, 5>$$

$$|\vec{v}_2(3)| = \sqrt{77}$$

Your answer: \text{Object A}
4. (10 points) Find an equation of the plane that passes through the point \((-1, 2, 1)\) and contains the line of intersection of the planes \(x + y - z = 2\) and \(2x - y + 3z = 1\).

Find 2 points on the line of intersection. Together with \(C(-1, 2, 1)\), these 3 points form a plane.

By inspection, we find 2 points on the line of intersection: \(A(1, 1, 0)\) and \(B(0, \frac{7}{2}, \frac{3}{2})\).

\[
\vec{n} = \vec{CA} \times \vec{CB} = \begin{vmatrix} i & j & k \\ 2 & -1 & -1 \\ 1 & \frac{3}{2} & \frac{1}{2} \end{vmatrix} = \langle 1, -2, 4 \rangle
\]

Thus, \(1(x+1) - 2(y-2) + 4(z-1) = 0\)

Your answer: \(x - 2y + 4z = -1\)
5. (10 points) Find an equation of the tangent plane to the surface given by the equation \( x^2 + y^2 - z^2 = 12 \) at point (2, 3, 1).

\[
\hat{f}(x, y, z) = x^2 + y^2 - z^2.
\]

The surface is a level surface of \( f \).

\[
\nabla f = < 2x, 2y, -2z >.
\]

\[
\nabla f(2, 3, 1) = < 4, 6, -2 >.
\]

Tangent plane is \( 4(x-2) + 6(y-3) - 2(z-1) = 0 \).

Your answer: \( 2x + 3y - z = 12 \).

Question 6. (1) (b) (2) (g) (3) (A)
6. (3) The correct answer is actually \(\sin(2x)\sin(2y)\): the picture has incorrect labels.
7. (10 points) There is a function \( f(x, y) \) which is differentiable whose exact formula is not known. Suppose, however, that we know that the intersection of the surface \( z = f(x, y) \) and the plane \( x = 1 \) is given by the curve \( \vec{r}_1(t) = (1, 1 + t + t^2, 1 - 2t) \), and the intersection of the surface \( z = f(x, y) \) and the plane \( y = 1 \) is given by the curve \( \vec{r}_2(t) = (t^3, 1, 2 - t) \). We also know that \( f(1, 1) = 1 \). Find the partial derivatives \( f_x(1, 1) \) and \( f_y(1, 1) \) evaluated at \( (x, y) = (1, 1) \).

\[
\begin{align*}
\vec{r}_1(t) &= (1, 1, 1) \quad \text{at} \quad t = 0 \\
\vec{r}_1'(t) &= <0, 1 + 2t, -2> \\
\vec{r}_1(0) &= <0, 1, -2> \\
\Rightarrow \quad \frac{\partial f}{\partial y}(1, 1) &= -2 \\
\end{align*}
\]

\[
\begin{align*}
\vec{r}_2(t) &= (1, 1, 1) \quad \text{at} \quad t = 1 \\
\vec{r}_2'(t) &= <3t^2, 1, 0, -1> \\
\vec{r}_2(1) &= <3, 0, 0> \\
\Rightarrow \quad \frac{\partial f}{\partial x}(1, 1) &= -\frac{1}{3}
\end{align*}
\]

Your answer:

\[
\begin{align*}
\frac{f_x}{x}(1, 1) &= -\frac{1}{3} \\
\frac{f_y}{y}(1, 1) &= -2
\end{align*}
\]
9. (10 points) Consider a right circular cylinder. The radius of the right circular cone is increasing at a rate of 2 cm/s while its height is decreasing at a rate of 3 cm/s. At what rate is the volume of the cylinder changing when the radius is 20 cm and the height is 30 cm?

\[ V = \pi r^2 h \]

\[ \frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt} \] by chain rule

At \[ r = 20, \ h = 30, \ \frac{dr}{dt} = 3, \ \frac{dh}{dt} = -3 \],

\[ \frac{dV}{dt} = 2\pi (20)(2)(30) + \pi (20^2)(-3) = 1200\pi \]

The rate of change is \[ 1200\pi \text{ cm}^3/\text{s} \]