Problem 1 (5 x 4 = 20 pts.): Multiple choice. No partial credit. Circle the best answer.
For this problem, consider the three planes described below. We will refer to them throughout the 4 parts of the problem (including on the next page).

Plane A: \( x - y = 2z + 1 \).
Plane B: \( x - z = \frac{3 - y}{2} \).
Plane C: \( x = y + 2z \).

i) Which of the three planes are parallel?

a) A and B only
b) B and C only
c) A and C only
d) All of them
e) None of them

\( \mathbf{n}_A = <1, -1, -2> \)
\( \mathbf{n}_B = <1, \frac{1}{2}, -1> \)
\( \mathbf{n}_C = <1, -1, -2> \)

ii) What is the (shortest) distance from the origin to Plane A?

a) 1
b) \( \frac{1}{\sqrt{6}} \)
c) \( \frac{1}{\sqrt{3}} \)
d) \( \frac{1}{\sqrt{6}} \)
e) None of the above
iii) Which of the following lines is completely contained in Plane C?

a) \( \mathbf{r}(t) = (t + 1, 1 - t, -2t) \)

b) \( x = 2t, \ \ y = 1, \ \ z = t \)

c) \( \frac{1}{3} = y = z \)

d) All of them

e) None of them

\[ \mathbf{u} = \langle 1, -1, -2 \rangle, \ \mathbf{v} = \langle 1, -1, -2 \rangle \]
\[ \langle 2, 0, 1 \rangle \cdot \langle 1, -1, -2 \rangle = 2 - 2 = 0 \ \checkmark \]
\[ \langle 3, 1, 1 \rangle \cdot \langle 1, -1, -2 \rangle = 3 - 1 - 2 = 0 \ \checkmark \]

iv) The osculating plane to the curve given by the vector valued function
\[ \mathbf{r}(t) = (\cos(t), (t - 1)^2, -\sin(t)) \]
at the point corresponding to \( t = 0 \) is ____________

a) Plane A

b) Plane B

c) Plane C

d) All of them

e) None of them

\[ \mathbf{r}'(t) = \langle -\sin(t), 2(t - 1), -\cos(t) \rangle \]
\[ \mathbf{r}''(t) = \langle -\cos(t), 2, \sin(t) \rangle \]
\[ \mathbf{r}'(0) = \langle 0, -2, -1 \rangle \]
\[ \mathbf{r}''(0) = \langle -1, 2, 0 \rangle \]
\[ \mathbf{r}'(0) \times \mathbf{r}''(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & -1 \\ -1 & 2 & 0 \end{vmatrix} = \langle 2, 1, -2 \rangle \]

Plane B?

\[ \mathbf{r}(0) = \langle 1, 1, 0 \rangle \]
\[ \sqrt{3 - 1} \]
\[ 1 - 0 = \frac{3 - 1}{2} \]
Problem 2 (5 x 3 = 15pts.): Multiple choice. No partial credit. Circle the best answer. Consider the points in space $A(2, 0, 0), B(2, -1, 1), C(3, 0, 1),$ and $D(1, 2, -1)$.

i) The area of the parallelogram spanned by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is

a) $1$

b) $\sqrt{2}$

c) $\sqrt{3}$

d) $\sqrt{6}$

e) None of the above

\[ \overrightarrow{AB} = \langle 0, -1, 1 \rangle; \quad \overrightarrow{AC} = \langle 1, 0, 1 \rangle \]

\[ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} + \mathbf{k} \]

\[ \| \overrightarrow{AB} \times \overrightarrow{AC} \| = \sqrt{(-1)^2 + (-1)^2 + 1^2} = 1 \sqrt{3} = \sqrt{3} \]

ii) Are the points $A, B, C,$ and $D$ coplanar?

a) Yes

b) No

c) It's simply impossible to know.

\[ \overrightarrow{AD} = \langle -1, 2, -1 \rangle \]

\[ \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \langle -1, 2, -1 \rangle \cdot \langle -1, 1, 1 \rangle = 1 \cdot 2 - 1 \cdot 1 = 2 \neq 0 \]

iii) Let $(x, y, z)$ be a vector that is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Which of the following must be true?

a) $x = y$

b) $x = z$

c) $y = z$

d) All of the above must be true.

e) None of the above must be true.

\[ \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{vmatrix} \]

\[ = \mathbf{i} (-1) - \mathbf{j} (0) + \mathbf{k} (2) = \langle -1, 0, 2 \rangle \]

\[ x = y = 2 \]
Problem 3 (5 \times 3 = 15 pts.): Multiple choice. No partial credit. Circle the best answer.  
Consider the function \( f(x, y) = 6x + 2y^3 + 3y^2 - 3x^2. \)

i) The point (1, -1) corresponds to

a) A local minimum of \( f \)

\[ \nabla f(x,y) = \begin{pmatrix} 6 - 6x & 6y^2 + 6y \end{pmatrix} \]

\[ \begin{cases} 6 - 6x = 0 \\ 6y^2 + 6y = 0 \end{cases} \]

Critical pt.: (1, 0); (1, -1)

\[ f_{xx}(x,y) = -6; \quad f_{yy}(x,y) = 12y + 6 \]

\[ f_{xy}(x,y) = 0 \]

\[ D(x,y) = -6(12y + 6) - 0^2 \]

\[ D(1,0) = -6(6) = -36 < 0 \]

\[ \Rightarrow \text{saddle pt.} \]

\[ D(1,-1) = -6(-12 + 6) = -6(-6) = 36 > 0 \]

\[ f_{xx}(1,-1) = -6 < 0 \]

\[ \Rightarrow \text{local max} \]

b) A local maximum of \( f \)

c) A saddlepoint of \( f \)

d) None of the above

e) It is impossible to tell from the information given.

ii) The point (0, -1) corresponds to

a) A local minimum of \( f \)

\[ f_{xx}(x,y) = 6; \quad f_{yy}(x,y) = 12y + 6 \]

\[ f_{xy}(x,y) = 0 \]

\[ D(x,y) = 6(12y + 6) - 0^2 \]

\[ D(0,-1) = 6(-6) = -36 < 0 \]

\[ \Rightarrow \text{saddle pt.} \]

\[ \text{D}(1,0) = -6(6) = -36 < 0 \]

\[ f_{xx}(1,0) = -6 < 0 \]

\[ \Rightarrow \text{local min} \]

b) A local maximum of \( f \)

c) A saddlepoint of \( f \)

d) None of the above

e) It is impossible to tell from the information given.

iii) The point (1, 0) corresponds to

a) A local minimum of \( f \)

\[ f_{xx}(x,y) = -6; \quad f_{yy}(x,y) = 12y + 6 \]

\[ f_{xy}(x,y) = 0 \]

\[ D(x,y) = -6(12y + 6) - 0^2 \]

\[ D(1,0) = -6(6) = -36 < 0 \]

\[ \Rightarrow \text{saddle pt.} \]

\[ D(1,-1) = -6(-12 + 6) = -6(-6) = 36 > 0 \]

\[ f_{xx}(1,-1) = -6 < 0 \]

\[ \Rightarrow \text{local max} \]

c) A saddlepoint of \( f \)

d) None of the above

e) It is impossible to tell from the information given.
The above figure shows some level curves for the function $g(x, y)$ in dark ink. In light ink are three planar unit vectors $a$, $b$ and $c$, which have been drawn at the point $(1, 9)$.

i) In the above picture, please mark clearly the location of any and all critical points for the function $g(x, y)$. If the critical point is a local maximum, mark it with the letter T. If the critical point is a local minimum, mark it with the letter B. If the critical point is a saddle point, mark it with the letter S.

ii) Of the directional derivatives $D_ag(1, 9)$, $D_bg(1, 9)$, and $D_cg(1, 9)$, determine which is the least, which is the greatest, and which has a value in between the other two. Circle one of the choices (the greatest / the least / in between) for each statement below, and give some justification for each of your choices.

- $D_ag(1, 9)$ is (the greatest / the least / in between) because ________________________________
- $D_bg(1, 9)$ is (the greatest / the least / in between) because ________________________________
- $D_cg(1, 9)$ is (the greatest / the least / in between) because ________________________________
Problem 5 (20 pts.) Today, Bruno doesn’t feel like doing anything. He has no responsibilities will just lounge around by the pool all day. The level of stress that he feels is a function of 3 things: his hunger level \( H \), the temperature outside \( T \), and the volume \( V \) at which his neighbor is playing heavy metal music. In terms of these three quantities, his stress level is given by the function

\[
S(H, T, V) = (V + 10)e^{-1/H} + (T - 80)^2 + V^3.
\]

i) What is the gradient of the function \( S(H, T, V) \) at the point \((1, 81, 3)\)?

\[
\nabla S(1, 81, 3) = \left< \frac{(V+10)}{H^2} e^{-1/H}, 2(T-80), e^{-1/H} + 3V^2 \right>
\]

ii) Let \( L(H, T, V) \) be the linear approximation of the function \( S(H, T, V) \) at the point \((1, 81, 3)\). Use your answer to part (i) to write a formula for \( L(H, T, V) \).

\[
L(H, T, V) = 13e^{-1} + 1 + 27 + 13e^{-1}(H-1) + 2(T-81) + (e^{-1} + 27)(V-3),
\]
iii) Let $t$ be the number of hours that Bruno has been sitting at the pool. Suppose that $H$, $T$, and $V$ as functions of $t$ are given by the formulas

$$H(t) = \frac{1+3t}{4}, \quad T(t) = 74 - t^2 + 8t, \quad V(t) = 3 + 3\sin(3(t-1)).$$

How fast is Bruno's stress level changing at the moment $t = 1$?

$$t = 1 \implies (H, T, V) = (1, 81, 3)$$

$$\frac{dH}{dt} = \frac{3}{4}, \quad \frac{dT}{dt} = -2t + 8; \quad \frac{dV}{dt} = 9\cos(3(t-1))$$

$$\left.\frac{dH}{dt}\right|_{t=1} = \frac{3}{4}; \quad \left.\frac{dT}{dt}\right|_{t=1} = 6; \quad \left.\frac{dV}{dt}\right|_{t=1} = 9$$

$$\frac{dS}{dt} = \nabla S(1, 81, 3) \cdot \begin{pmatrix} \frac{dH}{dt} \\ \frac{dT}{dt} \\ \frac{dV}{dt} \end{pmatrix}|_{t=1}$$

$$= \left\langle \frac{13}{e}, 12, \frac{1}{e} + 27 \right\rangle \cdot \left\langle \frac{3}{4}, 6, 9 \right\rangle$$

$$= \frac{39}{4e} + 12 + \frac{9}{e} + 3^5$$

$$= \frac{39 + 36}{4e} + 12 + 243$$

$$= \frac{75}{4e} + 255$$
Problem 6 (20 pts.) Consider the parabolic cylinder
\[ x = y^2 \]
and the hyperbolic paraboloid
\[ z = x^2 - y^2. \]

i) Write parametric equations for the curve formed by the intersection of the surfaces described above.

Let \( y = t \) \( \Rightarrow \) \( x = t^2 \), \( z = x^2 - y^2 = t^4 - t^2 \)

\[
\begin{align*}
\{ & x = t^2 \\
& y = t \\
& z = t^4 - t^2 \\
& t \in (-\infty, \infty)
\end{align*}
\]

ii) Write, but do not evaluate, an integral expression for the length of the curve from part (i) between the points \((1, 1, 0)\) and \((1, -1, 0)\).

\[(1, 1, 0) \Rightarrow t = 1; \quad (1, -1, 0) \Rightarrow t = -1 \]

\[
x'(t) = 2t \\
y'(t) = 1 \\
z'(t) = 4t^3 - 2t
\]

\[L = \int_{-1}^{1} \sqrt{(2t)^2 + 1 + (4t^3 - 2t)^2} \, dt\]
iii) Write parametric equations for the tangent line to the curve from part (i) at the point (1, 1, 0).

\[ \vec{r}(t) = \langle t^2, t, t^2 - t^2 \rangle \]
\[ \vec{r}'(t) = \langle 2t, 1, 4t^3 - 2t \rangle \]
\[ \vec{r}'(1) = \langle 2, 1, 2 \rangle \]

Line contaning \((1, 1, 0)\) in direction of \(\langle 2, 1, 2 \rangle\):

\[
\begin{align*}
X(s) &= 1 + 2s \\
y(s) &= 1 + s \\
z(s) &= 2s
\end{align*}
\]