1. (5 points) Find an equation of the sphere centered at the point (4, 1, 2) and is tangential to the plane \( x - 2y + 2z - 3 = 0 \).

Distance of point \((4,1,2)\) to the plane \( x - 2y + 2z - 3 = 0 \) is:

\[
\frac{|(4) - 2(1) + 2(2) - 3|}{\sqrt{1^2 + (-2)^2 + 2^2}} = \frac{3}{3} = 1
\]

Sphere of radius 1 centered at \((4,1,2)\) is:

\[(x - 4)^2 + (y - 1)^2 + (z - 2)^2 = 1\]

The equation is: \((x - 4)^2 + (y - 1)^2 + (z - 2)^2 = 1\)
2. (5 points) Let \( f(x, y) \) be a differentiable function. Suppose that the rate of change of \( f \) at the point \( P(1, 2) \) in the direction from \( P \) to \( Q(0, 1) \) is \( \sqrt{2} \). Suppose furthermore that the directional derivative \( D_{\vec{u}}f(1, 2) = -1 \) where \( \vec{u} = \frac{1}{2} \vec{i} - \frac{\sqrt{3}}{2} \vec{j} \). Find the partial derivatives \( f_x \) and \( f_y \) at \( P(1, 2) \).

\[
D_{\vec{u}} f = \vec{u} \cdot \nabla f = \left< f_x, f_y \right> \cdot \vec{u}
\]

For \( \vec{u} = \left< \frac{1}{2}, -\frac{\sqrt{3}}{2} \right> \), \(-1 = D_{\vec{u}} f = \left< f_x, f_y \right> \cdot \left< \frac{1}{2}, -\frac{\sqrt{3}}{2} \right> \)

\[
\Rightarrow -1 = \frac{1}{2} f_x - \frac{\sqrt{3}}{2} f_y \hspace{1cm} \text{at } P.
\]

Rate of change from \( P \) to \( Q \) is \( D_{\vec{v}} f(P) \)

\[
\text{Where}\quad \vec{v} = \frac{\left< 0, 1 \right> - \left< 1, 2 \right>}{\text{length}} = \frac{\left< -1, -1 \right>}{\text{length}} = \frac{\left< -1, -1 \right>}{\sqrt{2}}.
\]

\[
\sqrt{2} = D_{\vec{v}} f = \left< f_x, f_y \right> \cdot \frac{\left< -1, -1 \right>}{\sqrt{2}} = -\frac{1}{\sqrt{2}} f_x - \frac{1}{\sqrt{2}} f_y
\]

\[
\Rightarrow -1 = f_x + f_y
\]

Multiply (1) by 2 and add (2) \( \Rightarrow 0 = -\left( \sqrt{3} + 1 \right) f_y \)

\[
\Rightarrow f_y = 0.
\]

Hence \( f_x = -2 \)

\[
f_x(2, 1) = -2 \quad f_y(2, 1) = 0
\]
3. (5 points) Consider the function \( f(x, y, z) = x^2 + y^2/2 + 2z^2 + 2xz \). Find all points on the level surface \( f(x, y, z) = 4 \) at which the tangent plane is parallel to the \( xy \)-plane.

\[ \nabla f = \langle 2x + 2z, y, 4z + 2x \rangle \]

\( \nabla f \) gives the normal vector to the tangent plane.

We want tangent plane to be parallel to the \( xy \)-plane.

Hence the normal vector should be parallel to \( \frac{1}{2} \hat{k} \).

\[ \Rightarrow \begin{align*}
2x + 2z &= 0, \\
y &= 0
\end{align*} \]

\[ \frac{i}{i} \text{-component \hspace{1cm} \frac{j}{j} \text{-component of } \nabla f} \]

Now \((x, y, z)\) is on the level surface \( f(x, y, z) = 4 \)

\[ \Rightarrow \begin{align*}
x^2 + \frac{y^2}{2} + 2z^2 + 2xz &= 4
\end{align*} \]

By using \( y = 0, \ z = -x \), we find \( x^2 = 4, \ x = \pm 2 \)

The points are \((2, 0, -2), \ (-2, 0, 2)\)
4. (5 points. No partial points) Find a parametric equation of the curve of the intersection of the paraboloid $z = 4x^2 + y^2$ and the cylinder $x = y^2$.

Set $y = t$. Then $x = y^2 = t^2$,

and $z = 4x^2 + y^2 = 4(t^2) + t^2 = 4t^4 + t^2$.

The equation is $x = t^2$, $y = t$, $z = 4t^4 + t^2$, $-\infty < t < \infty$. 
5. (5 points) Consider the vector \( \vec{v} = \langle 3, 1, 5 \rangle \). Is it possible to decompose this vector into the form \( \vec{v} = \vec{A} + \vec{B} \) where the vector \( \vec{A} \) is parallel to the line \( \vec{r}(t) = \langle -3, -1, -4 \rangle + t \langle 1, 2, 3 \rangle \) and the vector \( \vec{B} \) is perpendicular to the plane \( 2x + y + 2z = -10 \), respectively? If your answer is yes, then find \( \vec{A} \) and \( \vec{B} \). If your answer is no, explain why in detail.

Circle one: Yes \[\bigcirc\] No

\[ \vec{A} \text{ is parallel to vector } \langle 1, 2, 3 \rangle \]
\[ \Rightarrow \vec{A} = a\langle 1, 2, 3 \rangle \text{ for a scalar } a \]

\[ \vec{B} \text{ is parallel to vector } \langle 2, 1, 1 \rangle \text{ (normal vector to plane } 2x + y + 2z = -10 \text{) } \]
\[ \Rightarrow \vec{B} = b\langle 2, 1, 1 \rangle \text{ for a scalar } b \]

To satisfy \( \vec{v} = \vec{A} + \vec{B} \),

\[ \langle 3, 1, 5 \rangle = a\langle 1, 2, 3 \rangle + b\langle 2, 1, 1 \rangle \]

\[ \Rightarrow \begin{cases} 3 = a + 2b & - (1) \\ 1 = 2a + b & - (2) \\ 5 = 3a + 2b & - (3) \end{cases} \]

(3) - (1) \[ \Rightarrow 2 = 2a \Rightarrow a = 1 \]
Plug it in (1) \[ \Rightarrow b = 1 \]
But this does not satisfy \( (2) : 1 \neq 2(1) + (1) = 3 \)
6. (5 points) Consider the vector \( \mathbf{v} = (3, 1, 5) \). Is it possible to decompose this vector into the form \( \mathbf{v} = \mathbf{A} + \mathbf{B} \) where the vector \( \mathbf{A} \) is parallel to the plane \( x + y + z = 2016 \) and \( \mathbf{B} \) is perpendicular to the same plane? If your answer is yes, then find \( \mathbf{A} \) and \( \mathbf{B} \). If your answer is no, explain why in detail.

Circle one:  
\[ \boxed{\text{Yes}} \quad \text{No} \]

\[ \mathbf{B} = \text{proj}_\mathbf{n} \mathbf{v} \text{, where } \mathbf{n} \text{ is a normal vector to the plane } x + y + z = 2016 \]

We take unit vector \( \mathbf{n} = \frac{(1, 1, 1.7)}{\sqrt{3}} \).

\[ \mathbf{B} = \text{proj}_\mathbf{n} \mathbf{v} = (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} = \left( (3, 1, 5) \cdot \frac{(1, 1, 1.7)}{\sqrt{3}} \right) \frac{(1, 1, 1.7)}{\sqrt{3}} \]

\[ = \left( \frac{9}{\sqrt{3}} \right) \frac{(1, 1, 1.7)}{\sqrt{3}} = 3 \frac{(1, 1, 1.7)}{\sqrt{3}} \]

\[ \mathbf{A} = \mathbf{v} - \mathbf{B} = (3, 1, 5) - (3, 3, 3.7) = (0, -2, 2.7) \]

\[ \mathbf{A} = (0, -2, 2.7) \]
\[ \mathbf{B} = (3, 3, 3.7) \]
7. (5 points) Find the length of the curve \( \mathbf{r}(t) = (2t, t^2, \frac{1}{3}t^3) \), \( 0 \leq t \leq 1 \).

\[
\mathbf{r}'(t) = \langle 2, 2t, t^2 \rangle
\]

\[
\text{length} = \int_0^1 \sqrt{2^2 + (2t)^2 + (t^2)^2} \, dt
\]

\[
= \int_0^1 \sqrt{4 + 4t^2 + t^4} \, dt
\]

\[
= \int_0^1 \sqrt{(2 + t^2)^2} \, dt
\]

\[
= \int_0^1 (2 + t^2) \, dt
\]

\[
= \left. (2t + \frac{1}{3}t^3) \right|_0^1
\]

\[
= 2 + \frac{1}{3} \cdot 1 = \frac{7}{3}
\]

The length is \( \frac{7}{3} \).
8. (5 points. No partial points) Which of the following equations does the function \( z = f(x+t) + g(x-t) \) satisfy for all differentiable functions \( f(s) \) and \( g(s) \) in a single variable?

(a) \( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} = 0 \)
(b) \( \frac{\partial z}{\partial x} = \frac{\partial z}{\partial t} \)
(c) \( \frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial t^2} = 0 \)
(d) \( \frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial t^2} \)
(e) \( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0 \)
(f) \( \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2} \)

\[
\frac{\partial z}{\partial x} = f'(x+t) + g'(x-t), \quad \frac{\partial^2 z}{\partial x^2} = f''(x+t) + g''(x-t)
\]
\[
\frac{\partial z}{\partial t} = f'(x+t) - g'(x-t), \quad \frac{\partial^2 z}{\partial t^2} = f''(x+t) + g''(x-t)
\]

Circle one:  (a)  (b)  (c)  (d)  (e)  (f)
9. (5 points) Consider the two lines \( \vec{r}_1(t) = (1, 1, 0) + t(1, -1, 2) \) and \( \vec{r}_2(t) = (2, 0, 2) + t(-1, 1, 0) \). Is there a plane which contains both of these lines? If your answer is yes, then find an equation of the plane. If your answer is no, explain the reason in detail.

Circle one:  
- Yes
- No

The lines are not parallel since the line 1 is parallel to \( \vec{v}_1 = (1, -1, 2) \) and the line 2 is parallel to \( \vec{v}_2 = (-1, 1, 0) \).

The lines meet at:

\[
\begin{align*}
\langle 1, 1, 0 \rangle + t \langle 1, -1, 2 \rangle &= \langle 2, 0, 2 \rangle + s \langle -1, 1, 0 \rangle \\
\Rightarrow & \quad \begin{cases} 
1 + t = 2 - s \\
1 - t = s \\
2t = 2
\end{cases} \\
\Rightarrow & \quad \begin{cases} 
s = 0 \\
t = 1
\end{cases}
\end{align*}
\]

\( \Rightarrow \) the lines meet at point \( (2, 0, 2) \).

\[
\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & -1 & 2 \\
-1 & 1 & 0
\end{vmatrix}
= \langle -2, -2, 0 \rangle
\]

The plane perpendicular to \( \langle -2, -2, 0 \rangle \) passing \( \vec{r}(2, 0, 2) \)

\[
\begin{align*}
\vec{n} \cdot (\vec{r} - \vec{r}(2, 0, 2)) &= 0 \\
-2(x-2) - 2(y-0) + 0(z-2) &= 0 \\
-2x + 4 - 2y &= 0 \\
-2x + 4 - 2y &= 0 \\
\text{Thus } x + y &= 2
\end{align*}
\]
10. (5 points) Consider the two lines \( \vec{r}_1(t) = t(1, 1, 1) \) and \( \vec{r}_2(t) = (-1, 0, 0) + t(1, 2, 3) \). Is there a plane which contains both of these lines? If your answer is yes, then find an equation of the plane. If your answer is no, explain the reason in detail.

Circle one: Yes \( \bigcirc \) No

Line 1 is parallel to \( \vec{v}_1 = (1, 1, 1) \)

Line 2 is parallel to \( \vec{v}_2 = (1, 2, 3) \)

Thus, they are not parallel.

Do they meet?

\[ t(1, 1, 1) = (-1, 0, 0) + s(1, 2, 3) \]

\[ \Rightarrow \begin{cases} t = -1 + s \\ t = 2s \\ t = 3s \end{cases} \]

\[ 2s = 3s \Rightarrow s = 0 \Rightarrow t = 0 \]

But \( t = 0, s = 0 \) do not solve \( t = -1 + s \).

Hence there are no \( t, s, \) and thus

there is no intersection point of line 1 and line 2.

Two lines do not meet.

\( \vec{v}_1 \)

Hence, there is no plane that contains both lines.