1. A regular tetrahedron has four vertices and any two of those vertices are at the same distance from each other. The four faces of a regular tetrahedron are all equilateral triangles.

(a) (3 points) Sketch a regular tetrahedron.
(b) (3 points) What is the angle between any two edges of a regular tetrahedron?
(c) (4 points) The three vertices \((0, 0, 0), (1, 0, 0),\) and \((\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)\) all lie in the x-y plane and are the vertices of an equilateral triangle. Find a fourth vertex \((x, y, z)\) that together with the three given vertices forms a tetrahedron.

**Answer:** (b) \(\frac{\pi}{3}\). (c) \(\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \pm \frac{\sqrt{3}}{\sqrt{3}}\right)\).

2. Both parts of this question are about the same plane.

(a) (6 points) Find the equation of the plane containing the line \(x = 3t + 2, y = -2t, z = -2t - 1\) and the point \((-1, 0, -3)\).
(b) (4 points) Find the distance of the origin from the plane of part (a).

**Answer:** (a) \(2x + 6y - 3z = 7\). (b) 1.

3. Consider the space curve \(\mathbf{r}(t) = \frac{t}{\sqrt{2}}\mathbf{i} + \frac{t}{\sqrt{2}}\mathbf{j} + t^2\mathbf{k}\).

(a) (6 points) Find the integral that gives the length of the curve from \(t = -2\) to \(t = 2\).
(b) (2 points) Use the indefinite integral

\[
\int \sqrt{1 + x^2} \, dx = \frac{x\sqrt{1 + x^2}}{2} + \frac{1}{2} \log \left( x + \sqrt{1 + x^2} \right)
\]

to evaluate the length in part (a).
(c) (2 points) What is the name of the space curve?

**Answer:** (a) \(\int_{-2}^{2} \sqrt{1 + 4t^2} \, dt\). (b) We have

\[
\int_{-2}^{2} \sqrt{1 + 4t^2} \, dt = \frac{1}{2} \int_{-4}^{4} \sqrt{1 + x^2} \, dx = \int_{0}^{4} \sqrt{1 + x^2} \, dx.
\]

The answer can be any one of the following:

\[
2\sqrt{17} + \frac{1}{2} \log \left( 4 + \sqrt{17} \right)
\]
\[
2\sqrt{17} + \frac{1}{4} \log \left( \sqrt{17} + 4 \right) - \frac{1}{4} \log \left( \sqrt{17} - 4 \right)
\]
\[
2\sqrt{17} + \frac{1}{4} \log \left( 33 + 8\sqrt{17} \right)
\]
\[
2\sqrt{17} + \log \sqrt{4 + \sqrt{17}}
\]
\[
2\sqrt{17} + \frac{\text{asinh}(4)}{2}
\]

Numerically, the answer is 9.293567525 (rounded to 10 digits). (c) Parabola.

4. The space curve \(\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}\) is a helix. The space curve \(\mathbf{r}(t) = \cos 2t \mathbf{i} + \sin 2t \mathbf{j} + 2t \mathbf{k}\) is the same helix parametrized differently, with \(t\) replaced by \(2t\).
(a) (2 points) Suppose the position vector of a particle is given by \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} \) with \( t \) being time. Find its speed \( \left| \frac{d\mathbf{r}}{dt} \right| \).

(b) (2 points) Suppose the position vector of a particle is given by \( \mathbf{r}(t) = \cos 2t \mathbf{i} + \sin 2t \mathbf{j} + 2t \mathbf{k} \) with \( t \) being time. Find its speed.

(c) (3 points) Suppose a particle moves on the same helix with initial position \( \mathbf{r}(0) = \mathbf{i} + 0 \mathbf{j} + 0 \mathbf{k} \) with speed equal to \( 3\sqrt{2} \). Find its position \( \mathbf{r}(t) \) as a function of time \( t \).

(d) (3 points) Suppose a particle moves on the same helix with initial position \( \mathbf{r}(0) = \mathbf{i} + 0 \mathbf{j} + 0 \mathbf{k} \) with speed equal to \( v \). Find its position \( \mathbf{r}(t) \) as a function of time \( t \).

Answer: (a) \( \sqrt{2} \). (b) \( 2\sqrt{2} \). (c) \( \mathbf{r}(t) = \cos 3t \mathbf{i} + \sin 3t \mathbf{j} + 3t \mathbf{k} \). (d):

\[
\mathbf{r}(t) = \cos \left( \frac{vt}{\sqrt{2}} \right) \mathbf{i} + \sin \left( \frac{vt}{\sqrt{2}} \right) \mathbf{j} + \frac{vt}{\sqrt{2}} \mathbf{k}.
\]

5. The equation \( x^3 + y^3 + z^3 = 3xyz \) implicitly gives \( z \) as a function of \( x, y \) and is therefore a surface.

(a) (3 points) Find \( \frac{\partial z}{\partial x} \).

(b) (3 points) Find \( \frac{\partial z}{\partial y} \).

(c) (4 points) The point \( (1, 1, -2) \) lies on the surface. Find the equation of the plane that is tangent to the surface at that point.

Answer: (a) \( \frac{x^2 - yz}{xy - zx} \). (b) \( \frac{y^2 - xz}{xy - zx} \). (c) \( (z + 2) = (x - 1)(-1) + (y - 1)(-1) \).

6. Find the partial derivative \( \frac{\partial z}{\partial x} \) in both parts.

(a) (5 points) \( z = x^2 + y^2 \).

(b) (5 points) \( z = \cos(u + v), \ u = x^2 - y^2, \ v = x^2 + y^2. \)

Answer: (a) \( 2x \). (b) \( -\sin(2x^2)(4x) \).

7. Let \( \ell_1 \) be the line given by \( (x, y, z) = (2t, t, 2t) \) and let \( \ell_2 \) be the line given by \( (x, y, z) = (-t + 3, -2t - 3, 2t + 3) \).

(a) (1 point) Find the cross-product of the vectors \( 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \) and \( -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \).

(b) (3 points) Find a plane that contains \( \ell_1 \) whose normal vector is the same as the cross-product of part (a).

(c) (3 points) Similarly, find a plane that contains \( \ell_2 \) whose normal vector is the same as the cross-product of part (a). Next, find the distance between the two planes.

(d) (3 points) Find the point \( P \) on \( \ell_1 \) and the point \( Q \) on \( \ell_2 \) such that the distance \( PQ \) is minimum and the same as the answer to part (c).

Answer: (a) \( 6\mathbf{i} - 6\mathbf{j} - 3\mathbf{k} \). (c) The distance is 3. To find \( P \) and \( Q \), solve

\[
(2u, u, 2u) + v(2, -2, -1) = (-w + 3, -2w - 3, 2w + 3)
\]

for \( u, v, w \) to find \( u = 1, \ v = 1, \ \text{and} \ w = -1. \) Therefore, \( P \) is \( (2, 1, 2) \) (with \( u = 1) \) and \( Q \) is \( (4, -1, 1) \) (with \( w = -1) \).

8. Consider the paraboloid surface \( z = x^2 + y^2 \).
(a) (1 point) The base point of the surface is $(0, 0, 0)$ and its axis (of symmetry) is the line $(x, y, z) = (0, 0, t)$. Sketch the surface showing the base point and the axis.

(b) (2 points) Now suppose the paraboloid is rotated so that the base point remains the base point but the axis of symmetry is the line $(x, y, z) = (6t, 2t, 3t)$. Sketch the rotated paraboloid showing the base point and the axis.

(c) (7 points) Find the equation of the rotated paraboloid of part (b)

**Answer:** The unit vector along the new axis is

$$
v = \frac{6}{7}i + \frac{2}{7}j + \frac{3}{7}k.
$$

The scalar component of the point $(x, y, z)$ along that vector is

$$\frac{6x + 2y + 3z}{7}.
$$

Therefore, the equation of the rotated paraboloid is

$$\frac{6x + 2y + 3z}{7} = x^2 + y^2 + z^2 - \left(\frac{6x + 2y + 3z}{7}\right)^2.$$