MATH 215 – Fall 2004

SECOND EXAM

Show your work in this booklet.
Do NOT submit loose sheets of paper—They won’t be graded

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Some useful trigonometric identities:

\[
\sin^2 \theta + \cos^2 \theta = 1 \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad \sin 2\theta = 2 \sin \theta \cos \theta
\]

\[
\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2}
\]

Spherical coordinates:

\[
x = \rho \cos(\theta) \sin(\phi) \quad y = \rho \sin(\theta) \sin(\phi) \quad z = \rho \cos(\phi)
\]
Problem 1. (10+5 = 15 points)
This problem is about the iterated double integral
\[ I = \int_{0}^{2} \int_{\sqrt{y/2}}^{1} y \exp(x^5) \, dx \, dy \]

(a) Sketch the region of integration and change the order of integration. On your sketch give the equations of the boundary curves and please CIRCLE your answers.

(b) Evaluate \( I \) by computing the double integral of your answer to part (a).

Answer: (b)
Problem 2. (10+15=25 points)

(a) Using the method of Lagrange multipliers, find the area of the largest rectangle with pairs of sides parallel to the coordinate axes that can be inscribed in the ellipse $x^2 + 4y^2 = 1$. Also give the coordinates of the corner of the rectangle in the first quadrant.

Answer: (a)
(b) Find and classify all the critical points of the function

$$g(x, y) = \sin(x) \cos(y)$$

in the square \([-1, 4] \times [-1, 4]\). CIRCLE your answers.
Problem 3.  (5+5= 10 points)  The plot below depicts the curve whose equation in polar coordinates is
\[ r = 2 - \cos(\theta) \]

(a) Write an iterated double integral in polar coordinates whose numerical value equals the area enclosed by the curve. CIRCLE your answer.

(b) Evaluate your answer to (a).

Answer: (b)
Problem 4. (7+8=15 points) Let $E$ be the region in the first octant bounded by the surfaces $2y^2 + z^2 = 8$ and $x + y = 2$, and let $f(x, y, z)$ be a function whose domain contains $E$. Denote by

$$I = \iiint_E f(x, y, z) \, dV$$

the integral of $f$ over $E$.

(a) Set up an iterated triple integral equal to $I$ and of the form

$$I = \iiint f \, dz \, dx \, dy.$$

CIRCLE your answer.
(b) Set up an iterated triple integral equal to $I$ and of the form

$$I = \iiint f \; dx \; dy \; dz.$$  

CIRCLE your answer.
Problem 5. (10+5=15 points) The problem is to find the Cartesian coordinates \((\bar{x}, \bar{y}, \bar{z})\) of the centroid of the region \(E\) inside the sphere \(x^2 + y^2 + z^2 = R^2\), and between the planes \(z = R/2\) and \(z = R\).

(a) Give a formula for \(\bar{z}\) in terms of an explicit iterated triple integral. Make sure you identify all limits of integration in the integral, and CIRCLE your answer. You may use without justification the fact that \(\text{Vol}(E) = \frac{5}{24}\pi R^3\).
(b) Compute $(\bar{x}, \bar{y}, \bar{z})$.
Problem 6. (12+8=20 points) The graph below is a plot of some of the level curves of a function \( f \) in a rectangular region \( R \). Assume that the change of the values of the function between adjacent level curves is the same everywhere. The arrows point in the direction of \( \nabla f \) (at each point they represent a unit vector in the direction of \( \nabla f \)).

(1) On the graph, clearly identify the (approximate) locations of the critical points of \( f \) in this region. For each critical point, indicate if it is a local maximum, local minimum or a saddle.

(2) Clearly mark the (approximate) location of the points where this function attains its global maximum and global minimum over the rectangle \( R \).