Problem 2. (5 + 5 + 10 = 20 points) Some unrelated problems involving spherical and cylindrical coordinates.

(a) Write the integral

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{0}^{x} (x^2 + y^2) \, dz \, dx \, dy$$

in cylindrical coordinates (do not evaluate).

\[ r = \sqrt{x^2 + y^2} \]

\[ x = r \cos(\theta) \]

\[ y = r \sin(\theta) \]

\[ z = z \]

\[ dx \, dy \, dz = r \, dr \, d\theta \, dz \]

(b) Set up the integral to express the volume which lies inside the sphere centered at the origin of radius 19 and between the cones \( \phi = \pi/4 \) and \( \phi = \pi/6 \). Do not evaluate.

$$\int_{0}^{2\pi} \int_{\pi/6}^{\pi/4} \int_{0}^{19} r^2 \sin(\phi) \, dr \, d\phi \, d\theta$$
(c) A hemispherical birdbath (i.e., bowl) of radius 13 cm is filled to within 5 cm of its top by water. Find the volume of the water in the birdbath.

\[
\begin{align*}
\text{Volume} &= \int_{0}^{\frac{12}{2}} \int_{0}^{\pi} \int_{0}^{\sqrt{169 - r^2}} r \, d\theta \, dz \, dr \\
&= 2\pi \int_{0}^{12} \left( -5 - \left( -\sqrt{169 - r^2} \right) \right) r \, dr \\
&= 2\pi \left[ \left( -\frac{5}{2} \right) - \left( \frac{169 - r^2}{3} \right)^{3/2} \right]_{0}^{12} \\
&= 2\pi \left[ \left( -\frac{5}{2} \right) - \left( \frac{25}{3} \right)^{3/2} - \left( 0 - \left( \frac{69}{3} \right)^{3/2} \right) \right] \\
&= 2\pi \left( \frac{992}{3} \right) \text{ cm}^3.
\end{align*}
\]
Problem 3. (10+10 = 20 points) Throughout this problem $C_1$ is the curve parametrized by $r(t) = (2 \cos(t), 2 \sin(t), t)$ for $0 \leq t \leq 2\pi$.

(a) Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (y^3z, -xy^2z, -4x^2z)$.

Throughout this problem, we shall write $c$ for $\cos(t)$ and $s$ for $\sin(t)$. So,

$$\vec{r}(t) = \langle 2c, 2s, t \rangle$$

$$\vec{F}'(t) = \langle -2s, 2c, 1 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 8s^3t, -8cs^2t, -16c^2t \rangle = 8t \langle s^3, -cs^2, -2c^2 \rangle$$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= 8t \langle s^3, -cs^2, -2c^2 \rangle \cdot \langle -2s, 2c, 1 \rangle \, dt$$

$$= \int_0^{2\pi} 8t (-2s^4 - 2c^2s^2 - 2c^2) \, dt$$

$$= -16 \int_0^{2\pi} t \left( \frac{s^4 + c^2s^2 + c^2}{s^2} \right) \, dt$$

$$= -16 \int_0^{2\pi} t \, dt = -8 (2\pi^2) = -32\pi^2$$
(b) Evaluate \( \int_{C_1} (x^2 + y^2)z \, ds \).

\[
\vec{r}(t) = \langle x(t), y(t), z(t) \rangle
\]

\[
ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt
\]

\[
= \sqrt{(-2s)^2 + (2c)^2 + 1^2} \, \frac{dt}{4}
\]

\[
= \sqrt{5} \, dt
\]

So,

\[
\int_{C_1} (x^2 + y^2)z \, ds = \int_0^{2\pi} (4c^2 + 4s^2) \, t \cdot \sqrt{5} \, dt
\]

\[
= \sqrt{5} \int_0^{2\pi} 4 + t \cdot dt
\]

\[
= 4\sqrt{5} \left[ \frac{t^2}{2} \right]_0^{2\pi} = 2 \sqrt{5} \left( 2\pi \right)^2
\]

\[
= 8\sqrt{5} \pi^2
\]
Problem 4. (5+5=10 points) Let $C_2$ be any path from $(0,1)$ to $(\pi/4,3)$.
(a) Show that the line integral
\[ \int_{C_2} 2y^2 \cos(2x) \, dx + 2y \sin(2x) \, dy \]
is independent of $C_2$.

Let $\vec{F}(x,y) = \frac{2y^2 \cos(2x)}{p}, \frac{2y \sin(2x)}{q}$

\[ \int_{C_2} 2y^2 \cos(2x) \, dx + 2y \sin(2x) \, dy = \int_{C_2} P \, dx + Q \, dy \]

\[ = \int_{C_1} \vec{F} \cdot d\vec{s} \]
is enough to check that $\vec{F}$ is conservative.

But, $P_y = 4y \cos(2x) = Q_y$, so $\vec{F}$ is conservative.

(b) Evaluate

\[ \int_{C_2} 2y^2 \cos(2x) \, dx + 2y \sin(2x) \, dy \]

Let $f(x,y) = y^2 \sin(2x)$. Then $\nabla f = \vec{F}$ so,

the Fundamental Theorem for Line Integrals:

\[ \int_{C_2} P \, dx + Q \, dy = f(\pi/4,3) - f(0,1) = (3 - 0) \]

\[ = 3 \sin(\pi/2) - 1 \sin(0) \]

\[ = 3 \]

\[ = 3 \]
Problem 5. (2x5=10 points)

(a) Suppose that \( C \) is a positively oriented, piecewise smooth, simple closed curve in the plane and let \( D' \) be the region bounded by \( C \). Use Green’s theorem to show that the area of \( D' \) is given by

\[
\frac{1}{2} \oint_C x\,dy - y\,dx.
\]

Green’s theorem says:

\[
\oint_C P\,dx + Q\,dy = \iint_{D'} (Q_x - P_y)\,dA.
\]

In our case, \( P = -\frac{y}{2} \) and \( Q = \frac{x}{2} \), so \( Q_x - P_y = \frac{1}{2} - (-\frac{1}{2}) = 1 \).

Hence

\[
\text{Area} (D') = \iint_{D'} 1\,dA = \iint_{D'} (Q_x - P_y)\,dA = \oint_C P\,dx + Q\,dy
\]

\[
= \oint_C -\frac{y}{2}\,dx + \frac{x}{2}\,dy = \frac{1}{2} \oint_C x\,dy - y\,dx.
\]

(b) Use part (a) to find the area of a disk of radius 17. (Hint: writing \( \pi \cdot (17)^2 \) is worth zero points.)

\[
\vec{r}(t) = \langle 17\cos(t), 17\sin(t) \rangle
\]

\[
0 \leq t \leq 2\pi
\]

Area (Disk) = \[
\frac{1}{2} \oint_C x\,dy - y\,dx
\]

\[
= \frac{1}{2} \int_0^{2\pi} (17\cos(t))(17\cos(t))\,dt - (17\sin(t))(-17\sin(t))\,dt
\]

\[
= \frac{17^2}{2} \int_0^{2\pi} (\cos^2(t) + \sin^2(t))\,dt = \frac{2\pi \cdot (17^2)}{2} = \pi \cdot (17)^2
\]
Problem 6. \((5+8+2+5 = 20\) points) Let \(R_1\) denote the region in \(\mathbb{R}^2\) which lies inside the circle \(r = 4 \cos(\theta)\) and outside the circle \(r = 2\sqrt{2}\).

(a) Sketch \(R_1\). Observe that \(4 \cos(\theta) = 2\sqrt{2} \implies \cos(\theta) = \frac{\sqrt{2}}{2} \implies \theta = \frac{\pi}{4}, -\frac{\pi}{4}\).

(b) If \(R_1\) is a thin lamina with density \(\rho(x, y) = \frac{1}{\sqrt{x^2+y^2}}\), find the mass of the lamina.

\[
\text{mass} = \iint_{R_1} \rho(x, y) \, dA
\]
\[
= \int_{-\pi/4}^{\pi/4} \int_{-4 \cos(\theta)}^{4 \cos(\theta)} \left(\frac{1}{r}\right) r \, dr \, d\theta
\]
\[
= \int_{-\pi/4}^{\pi/4} \left(4 \cos(\theta) - 2\sqrt{2}\right) \, d\theta
\]
\[
= \left[ 4\sin(\theta) - 2\sqrt{2}\theta \right]_{-\pi/4}^{\pi/4}
\]
\[
= 4\sqrt{2} - \sqrt{2}\pi = \sqrt{2} (4 - \pi)
\]
(c) What is the y-coordinate for the center of mass of the lamina? (Hint: Don’t think too much!)

\[ \bar{y} = 0 \] by symmetry.

(d) Write down the integral(s) describing the x-coordinate for the center of mass of the lamina. Do not evaluate.

\[ \bar{x} = \frac{\iint_{R_1} x \rho \, dA}{\iint_{R_1} \rho \, dA} = \frac{\int_{\pi/4}^{\pi} \int_{-\pi/4}^{\pi/4} r \cos(\theta) \, dr \, d\theta}{\sqrt{2} (4 - \pi)} \]