MATH 215

MIDTERM II
ANSWERS AND GRADING GUIDE

This Exam contains 5 problems. The problems are worth 12 points each. Each part of a problem counts equally. On problems 3, 4 and 5 you can get partial credit. Hence explain yourself carefully on these problems.

NO CALCULATOR.
2 TWO-SIDED 3in. BY 5in. NOTECARD OK.
CHECK YOUR SECTION IN THE TABLE

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Problem 1. TRUE FALSE QUESTIONS. NO PARTIAL CREDIT. CIRCLE TRUE OR FALSE. IF YOU THINK A QUESTION DOESN’T MAKE SENSE, CIRCLE FALSE

(a) \[ \int \frac{1}{1-x^2} \, dx = \arctan(-x) + C \quad TRUE/FALSE \]
ANSWER: FALSE

(b) \[ \int \cos(\cos x) \, dx = -\frac{\sin(\cos x)}{\sin x} + C \quad TRUE/FALSE \]
ANSWER: FALSE

(c) The double integral of the function \( f(x, y) = x^2 - y \) over the triangle with corners \((0,0), (1,0), (0,1)\) is equal to
\[ \int_0^1 \int_0^1 (-y + x^2) \, dy \, dx. \quad TRUE/FALSE \]
ANSWER: FALSE

(d) Suppose that \( \vec{F} \) and \( \vec{G} \) are conservative vector fields on \( \mathbb{R}^3 \). Then \( \vec{F} + \vec{G} \) is also conservative. TRUE/FALSE
ANSWER: TRUE: \( \vec{F} = \nabla f, \vec{G} = \nabla g, \vec{F} + \vec{G} = \nabla (f + g) \)

(e) The integral of the function \( f(x, y) = x^2y^2 \) over the disc \( x^2 + y^2 \leq 1 \) is equal to
\[ 4 \int_0^{\sqrt{1-x^2}} \int_0^1 x^2y^2 \, dx \, dy \quad TRUE/FALSE \]
ANSWER: FALSE

(f) The area of the disc \( x^2 + y^2 \leq 1 \) is equal to \( \int_C y \, dx + 2x \, dy \) where \( C \) is the circle \( x^2 + y^2 = 1 \) counterclockwise. TRUE/FALSE

ANSWER: TRUE, by GREENE’s THEOREM

WORKSPACE:
Problem 2. NO PARTIAL CREDIT
(a) The point \((x, y) = (-2, 2)\) Find the corresponding coordinates \((r, \theta)\).
(b) The point \((r, \theta) = (2, -2\pi/3)\). Find \((x, y)\).
(c) The point \((\rho, \theta, \phi) = (2, \pi/4, \pi/3)\). Find the corresponding coordinates \((x, y, z)\).
(d) The point \((x, y, z) = (0, 2\sqrt{3}, -2)\). Find the corresponding coordinates \((\rho, \theta, \phi)\).
(e) The point \((r, \theta, z) = (2, 2\pi/3, 1)\). Find \((x, y, z)\).
(f) \((x, y, z) = (3, -3, -7)\). Find \((r, \theta, z)\).

ANSWERS:
a: \((2\sqrt{2}, 5\pi/4)\)
b: \((-1, -\sqrt{3})\)
c: \((\sqrt{3}/2, \sqrt{3}/2, 1)\)
d: \((4, \pi/2, 5\pi/6)\)
e: \((-1, \sqrt{3}, 1)\)
f: \((3\sqrt{2}, -\pi/4, -7)\)

WORKSPACE:
Problem 3. **YOU CAN MAKE 0, 2 OR 4 POINTS ON EACH PART. 2 POINTS WILL ONLY BE AWARDED IN CASE OF A SMALL MISTAKE. THE THREE PARTS OF THE PROBLEM ARE UNRELATED**

(a) Evaluate

\[ \int \int_D dxdy \]

where \( D \) is the region bounded by \( x = y \) and \( x = y^2 - 2 \).

(b) Evaluate

\[ \int \int_D \sin(x^2 + y^2)dA \]

where \( D \) is the region between the two circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).

(c) Evaluate

\[ \int \int \int_E x(y^2 + z^2)dV \]

where \( E \) is the region bounded by the cylinder \( y^2 + z^2 = 1 \) and the planes \( x = 1, x = 2 \).

**ANSWERS:**

a: 4.5

b: \( \pi[\cos 1 - \cos 4] \)

c: \( 3\pi/4 \)

**WORKSPACE:**

(a) Limits of integration are \( y^2 - 2 \leq x \leq y \) and \( y^2 - 2 = y, -1 \leq y \leq 2 \).

\[
\int_{-1}^{2} \int_{y^2-2}^{y} dxdy = \int_{-1}^{2} (y - y^2 + 2)dy
\]

\[
= (y^2/2 - y^3/3 + 2y)_{-1}^{2}
\]

\[
= (2 - 8/3 + 4) - (1/2 + 1/3 - 2) = 4.5
\]
(b) \[
\int \int \sin(x^2 + y^2) \, dx \, dy = \int_0^{2\pi} \int_1^2 r \sin r^2 \, dr \, d\theta \\
= \int_0^{2\pi} \left[ \frac{-\cos r^2}{2} \right]_1^2 \, dr \\
= \int_0^{2\pi} \frac{1}{2} \left[ \cos 1 - \cos 4 \right] \, dr \\
= \pi \left[ \cos 1 - \cos 4 \right]
\]

(c) \[
\int \int \int_E x(y^2 + z^2) \, dV = \int \int (x^2/2) \, dy \, dz \\
= \frac{3}{2} \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta \\
= \frac{3}{2} \cdot 2\pi \cdot \frac{1}{4} = \frac{3\pi}{4}
\]
Problem 4. IF YOUR ANSWER IS CORRECT YOU GET FULL SCORE. IF ANSWER IS WRONG YOU CAN GET PARTIAL CREDIT IF YOU EXPLAIN YOURSELF CAREFULLY

(a) Evaluate
\[ \int \int \int_E (x^2 + y^2 + z^2)^5 \, dV \]
where \( E \) is the ball with center the origin and radius 2.

(b) Find the volume of the solid that lies above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere centered at the origin with radius 3.

(c) Find the gradient vector field of the function \( f(x, y, z) = xe^{xyz} \).

(d) Evaluate the line integral \( \int_C \vec{F} \cdot d\vec{r} \) where \( \vec{F} = <x, z, y^2> \) and \( \vec{r}(t) = <\sin t, t, t^3>, 0 \leq t \leq \pi/4. \)

ANSWERS:

a: \( 32768\pi/13 \)
b: \( 9\pi[2 - \sqrt{2}] \)
c: \( <e^{xyz} + xyz^2e^{xyz}, x^2ze^{xyz}, x^2yze^{xyz}> \)
d: \( 1/4 + \pi^4/4^5 + 3\pi^5/(4^5 \cdot 5) \)

WORKSPACE:

(a)
\[ \int \int \int_E (x^2 + y^2 + z^2)^5 \, dV \]
\[ = \int_0^2 \int_0^{2\pi} \int_0^\pi \rho^{10} \rho^2 \sin \phi d\phi d\theta d\rho \]
\[ = \int_0^2 \int_0^{2\pi} \rho^{12}(-\cos \phi)^\pi_0 d\theta d\rho \]
\[ = 2 \int_0^2 \int_0^{2\pi} \rho^{12} d\theta d\rho \]
\[ = 4\pi(\rho^{13}/13)_0^2 \]
\[ = 32768\pi/13 \]

(b)
\[ \int_0^3 \int_0^{2\pi} \int_0^{\pi/4} \rho^2 \sin \phi d\phi d\theta d\rho \]
\[ = \int_0^3 \int_0^{2\pi} \rho^2[1 - \sqrt{2}/2] d\theta d\rho \]
\[ = 9\pi[2 - \sqrt{2}] \]
(c) 

\[ \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/4} <\sin t, t^3, t^2 > \cdot <\cos t, 1, 3t^2> dt \]

\[ = \int_0^{\pi/4} \sin t \cos t + t^3 + 3t^4 dt \]

\[ = \left[ (\sin t)^2/2 + t^4/4 + 3t^5/5 \right]_0^{\pi/4} \]

\[ = 1/4 + \pi^4/4^5 + 3\pi^5/(4^5 \cdot 5) \]
Problem 5. IF YOUR ANSWER IS CORRECT YOU GET FULL SCORE. IF YOUR ANSWER IS WRONG YOU CAN GET PARTIAL CREDIT IF YOU EXPLAIN YOURSELF CAREFULLY

(a) Find the integral \( \int_C (2x + y)dx + (x + 2y)dy \) where \( C \) is the curve \( \vec{r}(t) = \langle \cos^3(t), e^{\sin(t)} \rangle \) with \( 0 \leq t \leq \pi \). [Hint: Write the integral as \( \int_C \vec{F} \cdot d\vec{r} \) and find \( f \) so that \( \nabla f = \vec{F} \).]

(b) Find \( \int_C (2y - e^{\arctan x})dx + (x^3 + \cos(sin(y)))dy \) where \( C \) is the boundary of the square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \).

ANSWERS:

a: \(-2\)
b: \(-1\)

WORKSPACE:

(a) \( f = x^2 + xy + y^2 \)
\( \int_C = f(1,1) - f(-1,1) = -2 \)

(b) Using Greene’s theorem:

\[
\int_0^1 \int_0^1 (3x^2 - 2)dydx = \int_0^1 3x^2 - 2dx
\]
\[
= 1 - 2 = -1
\]