1. (10 points) Find the volume below the surface \( z = x^4 + y^4 \) and above the square in the \( x-y \) plane with vertices at \((x, y) = (\pm 1, 0), (0, \pm 1)\) shown here:

![Diagram](image)

Solution: The volume is \( \frac{4}{15} \). The integration domain in the first quadrant is \( 0 \leq x \leq 1, \ 0 \leq y \leq 1 - x \). Thus, the answer is

\[
4 \int_0^1 \int_0^{1-x} x^4 + y^4 \, dy \, dx = 4 \int_0^1 x^4(1-x) + \frac{(1-x)^5}{5} \, dx \\
= 4 \left( \frac{1}{5} - \frac{1}{6} + \frac{1}{30} \right) \\
= \frac{4}{15}.
\]

2. Consider the iterated triple integral

\[
\int_0^1 \int_y^1 \int_0^y f(x, y, z) \, dz \, dx \, dy.
\]

In this integral, \( z \) is innermost, \( x \) is in the middle, and \( y \) is outermost.

(a) (5 points) Rewrite the integral with \( x \) innermost, \( y \) in the middle, and \( z \) outermost.

(b) (5 points) Rewrite the integral with \( z \) innermost, \( y \) in the middle, and \( x \) outermost.

Solution: The region of integration

\[
0 \leq y \leq 1 \\
y \leq x \leq 1 \\
0 \leq z \leq y
\]
can be described as $0 \leq z \leq y \leq x \leq 1$. For part (a), rewrite it as $0 \leq z \leq 1$, $z \leq y \leq 1$, and $y \leq x \leq 1$ to get

$$\int_0^1 \int_z^1 \int_y^1 f(x, y, z) \, dx \, dy \, dz.$$ 

Similarly, for part (b)

$$\int_0^1 \int_0^x \int_0^y f(x, y, z) \, dz \, dy \, dx.$$

3. Let $f(x, y) = x^4 + y^4 + 4xy$.

(a) (5 points) Find three critical points of $f(x, y)$.

(b) (5 points) Pick one of the three critical points and classify it as local minimum, local maximum, or saddle.

Solution: Set $f_x = 4x^3 + 4y = 0$ and $f_y = 4y^3 + 4x = 0$ to find that $(0, 0)$, $(1, -1)$, and $(-1, 1)$ are critical points. The matrix for the second derivative test is

$$\begin{pmatrix} 12x^2 & 4 \\ 4 & 12y^2 \end{pmatrix}.$$ 

The second derivative test implies that $(x, y) = (0, 0)$ is a saddle. That is because the determinant is $-16$ and negative. The other two critical points are both local minima (determinant and trace positive).

4. (10 points) Find a critical point (local maximum or minimum) of

$$f(x, y, z) = -x \log x - 2y \log y - 3z \log z$$

subject to the constraint

$$g(x, y, z) = x + 2y + 3z - 1 = 0.$$ 

Evaluate $f$ at that point. Here log is the natural logarithm, as usual, so that $\frac{d \log x}{dx} = \frac{1}{x}$.

Solution: Using the method of Lagrange multipliers, we get

$$-1 - \log x = \lambda$$

$$2(-1 - \log y) = 2\lambda$$

$$3(-1 - \log z) = 3\lambda,$$
or $x = y = z = e^{-1-\lambda}$. From the constraint equation $x + 2y + 3z = 1$, we get $x = y = z = 1/6$, which is therefore a critical point. The value of $f$ at the critical point is $\log 6$. In fact, the critical point is a maximum (but that was not asked in the problem).

5. (10 points) Find the two points at which the parabola $y = x^2/2$ intersects the circle $x^2 + y^2 = 8$. If $D$ is the region bounded by that parabola and circle (see below), evaluate the double integral:

$$
\int \int_D x^2 y \, dx \, dy.
$$

The region of integration $D$ looks as follows:

Note: Once you have a numerical answer you do not need to simplify it to a fraction. In the textbook, the area element $dx \, dy$ in the integral is given as $dA$.

Solution: The two points of intersection are $(x, y) = (\pm 2, 2)$. To evaluate the integral,

$$
\int_{-2}^{2} \int_{x^2/2}^{\sqrt{8-x^2}} x^2 y \, dy \, dx = \int_{-2}^{2} \frac{x^2}{2} \left( 8 - x^2 - \frac{x^4}{4} \right) \, dx \\
= \int_{-2}^{2} \left( 4x^2 - \frac{x^4}{4} - \frac{x^6}{8} \right) \, dx \\
= 2 \times \left( \frac{4 \cdot 2^3}{3} - \frac{2^5}{10} - \frac{2^7}{7.8} \right) \\
= \frac{1088}{105}.
$$

6. Consider the integral

$$
\int \int_D \frac{dx \, dy}{(x^2 + y^2)^{1/2}},
$$

where $D$ is the disc $(x - 1)^2 + y^2 \leq 1$.

(a) (4 points) Describe the region of integration in polar coordinates.
(b) (6 points) Evaluate the integral.

Solution: In polar coordinates, the region $D$ is given by $-\pi/2 \leq \theta \leq \pi/2$ and $0 \leq r \leq 2 \cos \theta$. Thus, the integral is equal to

$$\int_{-\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} r \ dr \ d\theta = \int_{-\pi/2}^{\pi/2} 2 \cos \theta \ d\theta = 4.$$

7. (10 points) Sketch the sector of the unit disc bounded by the lines $x = y$, $y = 0$, and the circle $x^2 + y^2 = 1$ in the first quadrant of the $x$-$y$ plane. Assuming constant density, find the $x$ and $y$ coordinates of the center of mass.

Solution: Use polar coordinates and describe $D$ as $0 \leq \theta \leq \pi/4$, $0 \leq r \leq 1$.

$$\bar{x} = \frac{\int \int_{D} x \ dx \ dy}{\int \int_{D} dx \ dy} = \frac{\int_{0}^{\pi/4} \int_{0}^{1} r \cos \theta \ r \ dr \ d\theta}{\int_{0}^{\pi/4} \int_{0}^{1} r \ dr \ d\theta} = \frac{\int_{0}^{\pi/4} \cos \theta / 3 \ d\theta}{\pi/8} = \frac{4 \sqrt{2}}{3 \pi}.$$

Thus, $\bar{x} = \frac{4 \sqrt{2}}{3 \pi}$. Similarly, $\bar{y} = \frac{4 \sqrt{2}(\sqrt{2} - 1)}{3 \pi}$. 

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