Second Midterm Exam: Problems and Answers

1. Consider the function \( f(x, y) = \sin(x^2 - y^2) \).
   a) (2 points) Find the gradient \( \nabla f \).
   b) (3 points) At the point \((x, y) = (1, 1)\), find the direction in which the directional derivative is maximum.
   c) (2 points) At the point \((x, y) = (1, 1)\), find the direction in which the directional derivative is zero.
   d) (3 points) Evaluate the limit
   \[
   \lim_{\delta \to 0} \frac{f(1 + \delta, 1 - 2\delta)}{\delta}.
   \]
   Answer: (a) \(2x \cos(x^2 - y^2)i - 2y \cos(x^2 - y^2)j\). (b) \(2i - 2j\) or more precisely \((i - j)/\sqrt{2}\). (c) \(i + j\) or more precisely \((i + j)/\sqrt{2}\). (d) 6.

2. Consider the double integral
   \[
   \int \int_D x^2y^2 \, dA.
   \]
   a) (6 points) Evaluate the integral if \(D\) is the triangular region with vertices \((0, 0)\), \((1, 0)\), and \((0, 1)\).
   b) (4 points) Evaluate the integral if \(D\) is the square with vertices at \((1, 0)\), \((0, 1)\), \((-1, 0)\), and \((0, -1)\).
   Answer: (a) \(1/180\) (b) \(1/45\).

3. Consider the solid sphere \(0 \leq x^2 + y^2 + z^2 \leq 1\). Suppose the density at a point in the solid is equal to \(1/r\), where \(r = \sqrt{x^2 + y^2 + z^2}\) (distance from the origin).
   a) (6 points) Find the total mass of the solid.
   b) (4 points) Find \(a > 0\) such that the mass inside the volume \(0 \leq x^2 + y^2 + z^2 \leq a^2\) is equal to the mass outside.
   Answer: (a) \(2\pi\). (b) \(a = 1/\sqrt{2}\).

4. Consider the quarter disc \(0 \leq x^2 + y^2 \leq 1\), \(x \geq 0\), \(y \geq 0\). Suppose the density (mass per unit area) at a point in the quarter disc is given by \(\rho(x, y) = x^2 + y^2\).
   a) (3 points) Find the total mass of the quarter disc.
   b) (4 points) Find \(\bar{x}\), the \(x\)-coordinate of the center of mass.
   c) (3 points) Find the distance of the center of mass from the origin.
   Answer: (a) \(\pi/8\). (b) \(\bar{x} = 8/3\). (c) \(8/3\sqrt{2}\).

5. Consider the paraboloid surface \(z = 2 - (x^2 + y^2)\) and the conical surface \(z^2 = x^2 + y^2\) with \(z \geq 0\).
   a) (3 points) Find the value of \(z\) at which the two surfaces intersect.
   b) (7 points) Find the surface area of the part of the paraboloid above that value of \(z\).
   Answer: (a) \(z = 1\). (b) \(\pi/6 \left(\sqrt{125} - 1\right)\).
6. The first part asks you to change the order of integration in a double integral and the second part in a triple integral.

a) (5 points) Consider the iterated double integral

\[ \int_0^1 \int_{y=x^3}^{x^2} f(x, y) \, dy \, dx, \]

which has \( y \) first (innermost) and \( x \) second. Sketch the region of integration and rewrite the iterated integral so that \( x \) is first (innermost) and \( y \) is second.

b) (5 points) Consider the triple integral

\[ \int_0^1 \int_0^z \int_x^1 f(x, y, z) \, dx \, dy \, dz, \]

which has \( x \) first (innermost), \( y \) second, and \( z \) third. Rewrite the iterated integral in reverse order: \( z \) first (innermost), \( y \) second, and \( x \) third.

**Answer:** (a):

\[ \int_0^1 \int_0^\sqrt{y} f(x, y) \, dx \, dy. \]

(b):

\[ \int_0^1 \int_0^{\min(x,y)} \int_x^z f(x, y, z) \, dx \, dy \, dz = \int_0^1 \int_0^x \int_0^{y} f(x, y, z) \, dz \, dy \, dx + \int_0^1 \int_0^x \int_z^x f(x, y, z) \, dz \, dy \, dx. \]

7. The first part is a maximization problem and the second part a minimization problem.

a) (5 points) Maximize the product \( xyz \) subject to the requirement \( 2x + y + 3z = 1 \).

b) (5 points) Find the point on the paraboloid \( z = x^2 + y^2 \) that is closest to the point \( P = (6, 3, 4) \).

**Solution:** (a) Use the Lagrange multiplier technique with \( f = xyz \) and \( g = 2x + y + 3z - 1 \) to get

\[ yz = 2\lambda \]
\[ zx = \lambda \]
\[ xy = 3\lambda. \]

After multiplying the above equations by \( x, y, z \), respectively, it follows that \( 2\lambda x = \lambda y = 3\lambda z \). Therefore, \( x = y/2 \) and \( z = y/3 \). Subbing into the equation \( 2x + y + 3z = 1 \), we find \( x = 1/6, y = 1/3 \), and \( z = 1/9 \). The maximum value of \( xyz \) is \( 1/162 \) for \( x, y, z \) positive. If \( x, y, z \) are allowed to be negative, the product can be unbounded and positive or negative.

(b) Choose \( f = (x - 6)^2 + (y - 3)^2 + (z - 4)^2 \) and \( g = x^2 + y^2 - z \) to get the equations

\[ (x - 6) = \lambda x \]
\[ (y - 3) = \lambda y \]
\[ 2(z - 4) = -\lambda. \]
Eliminate $\lambda$ between the first two equations to deduce $x = 2y$. Eliminate $\lambda$ between the second and third equations to deduce $z = 4 - \frac{y-3}{2y}$. Substitute into $z = x^2 + y^2$ to get the equation $5y^2 = \frac{7y+3}{2y}$ or $10y^3 - 7y - 3 = 0$. An obvious solution is $y = 1$. We also have $10y^3 - 7y - 3 = (y - 1)(10y^2 + 10y + 3)$, which shows that $y = 1$ is the only real solution. The closest point on the paraboloid is $(2, 1, 5)$. 