1. This problem has three parts.

(a) (2 points) Sketch the curves $z = 16 - x^2$ and $z = x^2 - 16$ in the $x$-$z$ plane. Determine the points of intersection of the two curves.

(b) (5 points) Find the area bounded by the curves $z = 16 - x^2$ and $z = x^2 - 16$ in the $x$-$z$ plane.

(c) (3 points) Find the volume bounded by the surfaces $z = 16 - x^2$, $z = x^2 - 16$, $y = 3$, and $y = -3$.

Answer: (a) Points of intersection: $x = \pm 4$, $z = 0$. (b) The area is

$$\int_{x=-4}^{x=4} \int_{z=x^2-16}^{z=16-x^2} dz \, dx = 4 \int_{0}^{4} (16 - x^2) \, dx = \frac{512}{3}.$$  

(c) Volume is 1024.

2. Let $f(x, y, z) = \frac{x^3+y^3+z^3}{3} - xyz$.

(a) (4 points) Find $\nabla f$ and evaluate it at $(x, y, z) = (1, 2, 3)$.

(b) (3 points) Find the unit vector $\mathbf{u}$ along which the directional derivative $D_\mathbf{u}f$ is maximum at $(x, y, z) = (1, 2, 3)$.

(c) (3 points) Find a unit vector $\mathbf{u}$ such that the directional derivative $D_\mathbf{u}f$ is zero.

Answer: (a) $\nabla f = \left( x^2 - yz, y^2 - xz, z^2 - xy \right)$. At the given point, it evaluates to $(-5, 1, 7)$. (b) $\mathbf{u} = \left( -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{7}{\sqrt{3}} \right)$. (c) $\mathbf{u} = (1, 5, 0)/\sqrt{26}$ for example.

3. Evaluate the triple integral $\iiint_B xyz \, dV$ for the following two choices of $B$:

(a) (5 points) $B$ is the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

(b) (5 points) $B$ is the region $0 \leq x \leq y \leq z \leq 1$.

Answer: (a) $\frac{1}{6}$. (b) $\frac{1}{48}$.

4. In both parts, the density is assumed to be $\rho(x, y) = x$.

(a) (5 points) The plot below

shows the sector of a circle of radius 1. The angle at the center is $\frac{\pi}{4}$ or $45^\circ$ as indicated. Find the mass of the sector.

(b) (5 points) In the plot below

a part of the sector is shaded. Find the mass of the shaded region.
Answer: (a) The mass is

\[ \int_0^{\pi/4} \int_0^1 r \cos \theta \, r \, dr \, d\theta = \frac{x^3}{3} \bigg|_0^{1/\sqrt{2}} \times \sin \theta \bigg|_0^{\pi/4} = \frac{1}{3 \sqrt{2}}. \]

(b) It is slightly easier to find the mass of the triangle that is not shaded.

\[ \int_0^{1/\sqrt{2}} \int_0^x x \, dy \, dx = \frac{x^3}{3} \bigg|_0^{1/\sqrt{2}} = \frac{1}{6 \sqrt{2}}. \]

The mass of the shaded region is \( \frac{1}{3 \sqrt{2}} - \frac{1}{6 \sqrt{2}} = \frac{1}{6 \sqrt{2}}. \)

5. In all the three parts, \( x \) and \( y \) are assumed to be positive with \( xy = 4 \).

(a) (3 points) Find the point \((x, y)\), with \( x, y > 0 \), that is an extremum of \( f(x, y) = \frac{x^2 + y^2}{2} \) subject to \( g(x, y) = xy = 4 \).

(b) (3 points) Sketch the curve \( g(x, y) = xy = 4 \) as well as the level curves of \( f(x, y) \) with \( f(x, y) \) being \( 1/2, 2, 8 \), respectively, in the first quadrant of the \( x-y \) plane. Is the extremum of part (a) a minimum or a maximum? Explain.

(c) (4 points) Find the minimum value of \( \frac{x^3}{3} + \frac{2y^{3/2}}{3} \) subject to \( xy = 4 \) and \( x, y \) positive.

Answer: (a) \( x = y = 2 \). (b) The extremum is a minimum. (c) The minimum value of 4 occurs at \( x^3 = y^{3/2} = 4 \).

6. Consider the surface given by \( x^2 + y^2 + 3z^2 = 3 \) and the plane \( x + y + z = 1 \).

(a) (3 points) Explain why the plane intersects the surface.

(b) (7 points) Suppose the curve along which the plane intersects the surface is \( C \). Find the maximum and minimum values of \( z \) for points on \( C \).

Answer: (a) The point \((1, 0, 0)\) lies on the plane and satisfies \( x^2 + y^2 + 3z^2 < 3 \). On the other hand, the point \((-1, -1, 3)\) also lies on the plane but satisfies \( x^2 + y^2 + 3z^2 > 3 \). Therefore, the plane must intersect the surface. Alternatively, observe that \((0, 0, 1)\) is a point of intersection of the plane and the surface. (b) On the plane, \( y = 1 - x - z \). Therefore, we want extrema of \( z \) subject to \( g(x, z) = x^2 + (1 - x - z)^2 + 3z^2 = 3 \). From \( \frac{\partial g}{\partial x} = 0 \), we get

\[ x = \frac{1 - z}{2}. \]

Substituting in \( g(x, z) = 0 \), we get

\[ \frac{(1 - z)^2}{2} + 3z^2 = 3, \]

which is solved by \( z = 1 \) and \( z = -5/7 \).

7. The surface \( x^2 + y^2 + z^2 = 1 \) is the sphere, and as you know, the area of the upper hemisphere with \( z > 0 \) is \( 2\pi \). In this problem, you will find areas of portions of that surface and your answers should therefore be less than \( 2\pi \).
(a) (4 points) Find the area of the upper hemisphere inside the cylindrical surface \( x^2 + y^2 = \frac{1}{4} \).

(b) (6 points) Find the area of the upper hemisphere inside the cylindrical surface \( (x - \frac{1}{2})^2 + y^2 = \frac{1}{4} \).

Answer: (a) Let \( D \) be the region \( x^2 + y^2 \leq \frac{1}{4} \) in the \( x-y \) plane. Because \( \frac{\partial z}{\partial x} = -\frac{x}{z} \) and \( \frac{\partial z}{\partial y} = -\frac{y}{z} \), the surface area is

\[
\int \int_D \frac{1}{z} \, dx \, dy = \int_{r=0}^{1/2} \int_{\theta=0}^{2\pi} \frac{1}{\sqrt{1 - r^2}} \, r \, dr \, d\theta = 2\pi \times -(1 - r^2)^{1/2} \bigg|_0^{1/2} = 2\pi \left( 1 - \frac{\sqrt{3}}{2} \right).
\]

(b) If \( D \) is \( (x - \frac{1}{2})^2 + y^2 \leq \frac{1}{4} \), its boundary is \( x^2 + y^2 = x \) or \( r = \cos \theta \). Thus the area of the surface is

\[
\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{\cos \theta} \frac{r}{\sqrt{1 - r^2}} \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} -(1 - (r^2)^{\cos \theta}) \bigg|_0^{\cos \theta} \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} (1 - |\sin \theta|) \, d\theta = 2 \int_0^{\pi/2} (1 - \sin \theta) \, d\theta = \pi - 2.
\]

It follows that the area of the sphere \( x^2 + y^2 + z^2 = 1 \) outside of the cylinders \( (x \pm \frac{1}{2})^2 + y^2 = \frac{1}{4} \) is \( 4\pi - 4(\pi - 2) = 8 \), an integer! This result is due to Vincenzo Viviani, a student of Galileo.