MATH 215 – Fall 2004

FINAL EXAM

Show your work in this booklet.
Do NOT submit loose sheets of paper–They won’t be graded

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Some useful trigonometric identities:

\[
\sin^2 \theta + \cos^2 \theta = 1
\]
\[
\sin^2 \theta = \frac{1 - \cos 2\theta}{2}
\]
\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta
\]
\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}
\]
\[
\sin 2\theta = 2 \sin \theta \cos \theta
\]

Spherical coordinates:

\[
x = \rho \cos(\theta) \sin(\phi)
\]
\[
y = \rho \sin(\theta) \sin(\phi)
\]
\[
z = \rho \cos(\phi)
\]
Problem 1.  (15 points) This problem is about the function

\[ f(x, y, z) = 3zy + 4x \cos(z). \]

(a) What is the rate of change of the function of \(f\) at \((1, 1, 0)\) in the direction from this point to the origin?

(b) Give an approximate value of \(f(0.9, 1.2, 0.11)\).
(c) Recall that $f(x, y, z) = 3zy + 4x \cos(z)$.
The equation $f(x, y, z) = 4$ implicitly defines $z$ as a function of $(x, y)$, if we agree that $z = 0$ if $(x, y) = (1, 1)$.
Find the numerical values of the derivatives
\[
\frac{\partial z}{\partial x}(1, 1) \quad \text{and} \quad \frac{\partial z}{\partial y}(1, 1).
\]
Problem 2. (10 points)
(a) Find and classify the critical points of the function $f(x, y) = -2x^2 + 8xy - 9y^2 + 4y - 4$.

(b) Find the equation of the tangent plane to the surface $z + 2x^2 - 8xy + 9y^2 - 4y = 0$ at the point $(2, 0, -8)$. 
Problem 3. (25 points) No partial credit. Evaluate each integral below. $R > 0$ and $a > 0$ are constants. Note: If you correctly understand the domains and/or the symmetries of the functions, many of the integrations become immediate. CIRCLE your answers.

(1) If $D = [-1, 0] \times [-1, 0] \cup [0, 1] \times [0, 1]$, then
$$\int \int_D xy \, dA =$$

(2) $\int_0^a \int_0^x dy \, dx =$

(3) If $D$ is defined in polar coordinates by the inequalities: $0 \leq r \leq R$, $\pi/7 \leq \theta \leq 8\pi/7$,
then $\int \int_D \sqrt{x^2 + y^2} \, dA =$

(4) $\int_0^{\pi/2} \int_0^\pi \int_0^R \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi =$

(5) $\int_0^{\pi/4} \int_0^R \int_0^a r \, dz \, dr \, d\theta =$
Problem 4. (10 points) (a) Find the value of $a$ such that the field on the plane

$$\vec{F}(x, y) = (axy + \frac{1}{x}, x^2)$$

is conservative. Find a potential for the resulting field.

(b) Compute the line integral of the conservative field you found in part (a) over the curve which is the image of $\langle e^{t^2}, t \cos(2\pi t) \rangle$, where $0 \leq t \leq 1$. 
Problem 5. (9+6=15 points) (a) By a direct calculation, evaluate \( I = \int_{C_R} dx + x^2 y dy \), where \( C_R \) is the triangle with vertices \((0,0), (0, R), (R, 0)\) oriented counterclockwise.

(b) Compute the value of \( I \) by evaluating a double integral, using Green’s theorem.
Problem 6. (15 points) Let $S$ be the portion of the surface $x = 5 - y^2 - z^2$ in the half space $x \geq 1$, oriented so that the normal vector at $(5, 0, 0)$ is equal to $\vec{i}$. Let $\vec{F}(x, y, z) = (-1, 1, 0)$ (a constant vector field).

(a) Set up and evaluate an iterated double integral equal to $\iint_S \vec{F} \cdot d\vec{S}$. 

\[ \int \int_S \vec{F} \cdot d\vec{S} \]
(b) It turns out that $\vec{F} = \nabla \times \vec{G}$, where $\vec{G} = \langle 0, z, -x \rangle$. (You do not have to verify this.) Give an alternative calculation of the surface integral of part (a) by applying Stokes' theorem.
Problem 7. (10 points) Consider the plot of a vector field \( \vec{F} = P(x, y)\hat{i} + Q(x, y)\hat{j} \):

(a) Mark a point \( A \) at which \( \text{curl}(\vec{F}) > 0 \).
(b) Mark a point \( B \) at which \( \text{div}(\vec{F}) < 0 \).
(c) Sketch a closed curve \( C \) such that the circulation of \( \vec{F} \) along \( C \) is positive.
(d) Mark two points \( S \) and \( T \) and two curves \( C_1 \) and \( C_2 \) from \( S \) to \( T \) such that the work done by \( \vec{F} \) in moving an object along those curves is positive for \( C_1 \) and negative for \( C_2 \).
(e) Can the field \( \vec{F} \) be a gradient field? On the space below explain why or why not.