Your name: ________________________________

Instructions:

• This examination booklet contains 10 problems on 12 sheets of paper including the front cover. The last page is left blank for your own use. The second last page contains a list of formulas.

• This is a 120-minute exam.

• This is a closed book exam. Calculators and note-cards are not allowed.

• Show your work and explain clearly. Otherwise, no points will be given.
Your scores:

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1. (10 points) Let $a$ be a number satisfying $0 < a < \pi$.

(a) Consider the ball $B$ of radius $R$ centered at the origin. The cone with opening angle $a$, given by the equation $\phi = a$, divides the ball into two solids. Let $B_a$ be the solid containing the “north pole” $(0, 0, R)$. (Hence when $a < \pi/2$, $B_a$ looks like an ice-cream cone with a spherical cap.) Evaluate the ratio of the volume of $B_a$ to the volume of $B$. (Recall that the volume of a ball of radius $R$ is $\frac{4}{3}\pi R^3$.)

The ratio is 

(b) Consider the sphere $S$ of radius $R$ centered at the origin. The cone $\phi = a$ divides the sphere into two surfaces. Let $S_a$ be the surface containing $(0, 0, R)$. Evaluate the ratio of the surface area of $S_a$ to the surface area of the whole sphere $S$. (Recall that the surface area of a sphere of radius $R$ is $4\pi R^2$.)

The ratio is 

2. (10 points) Let $C$ be the curve of intersection of the plane $x - z = 2$ and the cylinder $x^2 + y^2 = 1$. The curve is oriented counterclockwise when viewed from the above. Let

$$\vec{F}(x, y, z) = \langle -y + e^{-x^2}, x^2, -z^3 \rangle.$$ 

Evaluate the circulation $\oint_C \vec{F} \cdot d\vec{r}$ of $\vec{F}$ along $C$. 

The circulation is ________________________________
3. (10 points) Let
\[
\vec{F}(x, y, z) = (3x^2yz - 3y, x^3z - 3x, x^3y + 2z).
\]
Evaluate the work done by the force field \( \vec{F} \) in moving a particle along the following curve from point \((0, 0, 2)\) to point \((0, 3, 0)\).

The work is

\[\text{____________________________} \]
4. (10 points) Evaluate

\[ \oint_C xy \, dx + x^2 y^3 \, dy \]

where \( C \) is the positively oriented closed curve given in the picture.

The integral is __________________________
5. (10 points) (No partial points) (No need to explain) Each of the following four vector fields $\vec{F}(x, y, z)$ is shown in the $xy$-plane and looks the same in all other horizontal planes. In other words, $\vec{F}$ is independent of $z$ and its $z$-component is 0. Answer the following questions. You do not need to provide an explanation. No partial points are given for these questions.

(a) Find all points (among the 4 points A, B, C, D) at which $\text{div} \vec{F} \neq 0$.

The answer is ________________________________________________________________________

(b) Find all points (among the 4 points A, B, C, D) at which $\text{curl} \vec{F} \neq \vec{0}$.

The answer is ________________________________________________________________________

(c) Find all closed curves (among the 6 curves $C_1, C_2, \cdots, C_6$) along which the circulation $\oint_C \vec{F} \cdot d\vec{r}$ of the corresponding vector field is not zero.

The answer is ________________________________________________________________________

(d) For this question, regard the above vector fields as two-dimensional vector fields $\vec{G}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$. Find all closed curves (among the 6 curves $C_1, C_2, \cdots, C_6$) along which the flux $\oint_C \vec{G} \cdot \hat{n} \, ds$ of the corresponding vector field is not zero, where $\hat{n}$ is the outward unit normal vector.

The answer is ________________________________________________________________________
6. (10 points) Consider the vector field

\[ \vec{F}(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}. \]

This vector field satisfies \( \text{div} \vec{F}(x, y, z) = 0 \) for all \((x, y, z)\) except for \((0, 0, 0)\) at which the vector field is not defined. Let \( S \) be the closed surface given by the boundary of the box \(-2 \leq x \leq 2, -3 \leq y \leq 3, -4 \leq z \leq 4\). We orient \( S \) outward. Evaluate the flux of \( \vec{F} \) across \( S \).

The flux is ________________________________
7. (5 points) A particle starts at \((1, 1, 0)\) with initial velocity \(\langle 1, -1, 3 \rangle\). Its acceleration at time \(t\) is \((6t, 12t^2, -6t)\). Where is the particle at time \(t\)?

The particle is at _____________________________

8. (5 points) Let \(f(x, y)\) be a function which has continuous partial derivatives. Consider the points \(A(1, 1), B(3, 1), C(0, 2),\) and \(D(4, 4)\). Suppose that the directional derivative of \(f\) at \(A\) in the direction of the vector \(\overrightarrow{AB}\) is 3 and the directional derivative of \(f\) at \(A\) in the direction of the vector \(\overrightarrow{AC}\) is \(\frac{1}{\sqrt{2}}\). Find the directional derivative of \(f\) at \(A\) in the direction of the vector \(\overrightarrow{AD}\).

The directional derivative is ______________________________
9. (5 points) Suppose that the electric potential $V$ is given by $V(x, y, z) = 5x^2 - 3xy + xyz$. In which direction or directions does $V$ change most rapidly at $P(0, 1, 2)$?

$V$ changes most rapidly in the direction of the vector(s) ________________________________

10. (5 points) Find a parametric equation for the line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$.

The parametric equation is ________________________________
You may find some of the following formulas useful.

- \( \sin^2(x) + \cos^2(x) = 1 \), \( \cos(2x) = \cos^2(x) - \sin^2(x) \), \( \sin(2x) = 2 \sin(x) \cos(x) \)
- \( \sin(x) \cos(x) = \frac{1}{2} \sin(2x) \), \( \sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x) \), \( \cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x) \)
- \( \cos(\pi/3) = 1/2 \), \( \sin(\pi/3) = \sqrt{3}/2 \), \( \cos(\pi/4) = \sqrt{2}/2 \), \( \sin(\pi/4) = \sqrt{2}/2 \), \( \cos(\pi/6) = \sqrt{3}/2 \), \( \sin(\pi/6) = 1/2 \), \( \cos(0) = 1 \), \( \sin(0) = 0 \).
- \( \frac{d}{dx} \sin(x) = \cos(x) \), \( \frac{d}{dx} \cos(x) = -\sin(x) \).
- \( \text{proj}_b \mathbf{a} = \left( \frac{\mathbf{b} \cdot \mathbf{a}}{\| \mathbf{b} \|} \right) \mathbf{b} \), \( \text{comp}_b \mathbf{a} = \frac{\mathbf{b} \cdot \mathbf{a}}{\| \mathbf{b} \|} \)
- \( (\mathbf{a} \times \mathbf{b}) \times (\mathbf{d} \times \mathbf{f}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} = (bf - ce, -af + cd, ae - bd) \).
- \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \), \( \mathbf{a} \times (\mathbf{ab}) = (\mathbf{aa}) \times \mathbf{b} = \alpha (\mathbf{a} \times \mathbf{b}) \).
- Area of the parallelogram determined by the vectors \( \mathbf{v}_1 = (a, b, c) \) and \( \mathbf{v}_2 = (d, e, f) \) is \( |\mathbf{v}_1 \times \mathbf{v}_2| \).
- Volume of the parallelepiped determined by the vectors \( \mathbf{v}_1 = (a, b, c) \), \( \mathbf{v}_2 = (d, e, f) \), and \( \mathbf{v}_3 = (g, h, i) \) is \( |\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \).
- Distance from a point \( (a, b, c) \) to a plane \( Ax + By + Cz + D = 0 \) is \( \frac{|Aa + Bb + Cc + D|}{\sqrt{A^2 + B^2 + C^2}} \).
- Arc length function: Length of curve from \( r(\alpha) = (x(\alpha), y(\alpha), z(\alpha)) \) to \( r(t) = (x(t), y(t), z(t)) \) is \( s(t) = \int_a^t |r'(u)| du \).
- \( \mathbf{T}(t) = \frac{r'(t)}{|r'(t)|} \), \( \mathbf{N}(t) = \frac{r''(t)}{|r''(t)|} \)
- Second derivative test: \( D = f_{xx}f_{yy} - f_{xy}^2 \).
- The circumference of a circle of radius \( R \) is \( 2\pi R \). The area of a disk of radius \( R \) is \( \pi R^2 \). The volume of a ball of radius \( a \) is \( \frac{4\pi R^3}{3} \). The surface area of a sphere of radius \( R \) is \( 4\pi R^2 \). The volume of a cone with base radius \( a \) and height \( b \) is \( \frac{1}{3} \pi a^2 b \).
- Green’s Theorem: \( \oint_{\partial D} P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \).
- Stokes' Theorem: \( \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} \).
- Divergence Theorem: \( \iiint_E \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\nabla \cdot \mathbf{F}) \, dV \).
- For an oriented surface parametrized by \( \bar{r}(u, v) \), \( d\mathbf{S} = \mathbf{n} \, dA = (\bar{r}_u \times \bar{r}_v) \, dA \).
Scratch paper.