1. (10 points) Let \( a \) be a number satisfying \( 0 < a < \pi \).

(a) Consider the ball \( B \) of radius \( R \) centered at the origin. The cone with opening angle \( a \), given by the equation \( \varphi = a \), divides the ball into two solids. Let \( B_a \) be the solid containing the “north pole” \((0,0,R)\). (Hence when \( a < \pi/2 \), \( B_a \) looks like an ice-cream cone with a spherical cap.) Evaluate the ratio of the volume of \( B_a \) to the volume of \( B \). (Recall that the volume of a ball of radius \( R \) is \( \frac{4}{3}\pi R^3 \).)

Solution:

\[
\text{vol}(B_a) = \iiint_{B_a} dV = \int_0^{2\pi} \int_0^a \int_0^R \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{2\pi}{3} R^3 (1 - \cos \varphi).
\]

Since \( \text{vol}(B) = \frac{4}{3}\pi R^3 \), the ratio is \( \frac{1}{2}(1 - \cos \varphi) \).

(b) Consider the sphere \( S \) of radius \( R \) centered at the origin. The cone \( \varphi = a \) divides the sphere into two surfaces. Let \( S_a \) be the surface containing \((0,0,R)\). Evaluate the ratio of the surface area of \( S_a \) to the surface area of the whole sphere \( S \). (Recall that the surface area of a sphere of radius \( R \) is \( 4\pi R^2 \).)

Solution:

\[
\text{area}(S_a) = \iint_{S_a} dS = \int_0^{2\pi} \int_0^a R^2 \sin \varphi \, d\varphi \, d\theta = 2\pi R^2 (1 - \cos \varphi).
\]

Since \( \text{area}(S) = 4\pi R^2 \), the ratio is \( \frac{1}{2}(1 - \cos \varphi) \).
2. (10 points) Let $C$ be the curve of intersection of the plane $x - z = 2$ and the cylinder $x^2 + y^2 = 1$. The curve is oriented counterclockwise when viewed from the above. Let

$$\vec{F}(x, y, z) = \langle -y + e^{-x^2}, x^2, -z^3 \rangle.$$ 

Evaluate the circulation $\oint_C \vec{F} \cdot d\vec{r}$ of $\vec{F}$ along $C$.

**Solution:** The curve is an ellipse lying on the plane $x - z = 2$. Let $S$ be the flat surface on the plane bounded by this curve. Using Stokes' theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{S}$ where $d\vec{S}$ points upward (i.e. the $z$-component is positive) due to the choice of the orientation of $C$. Now, it is straightforward to compute $\text{curl} \vec{F} = (0, 0, 2x + 1)$. On the other hand, $d\vec{S}$ can be computed by using the fact that the surface is a part of the plane $x - z = 2$. The equation of the plane can be re-expressed as $z = g(x, y)$ where $g(x, y) = x - 2$. Since this is a graph, we know that $d\vec{S} = (1, 0, -1) dA$. Hence we find that $\text{curl} \vec{F} \cdot d\vec{S} = (2x + 1) dA$ and

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{S} = \iint_D (2x + 1) dA$$

where $D$ is the projection of $S$ onto the $xy$-plane. It is easy to see from the geometry that $D$ is the disk of radius 1 centered at the origin $(0, 0)$ on the $xy$-plane. Thus, using polar coordinates,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{S} = \iint_D (2x + 1) dA = \int_0^{2\pi} \int_0^1 (2r \cos \theta + 1) r dr d\theta = 2\pi.$$
3. (10 points) Let
\[ \vec{F}(x, y, z) = (3x^2yz - 3y, x^3z - 3x, x^3y + 2z). \]
Evaluate the work done by the force field \( \vec{F} \) in moving a particle along the following curve from point \((0, 0, 2)\) to point \((0, 3, 0)\).

**Solution:** Note that work = \( \int_C \vec{F} \cdot d\vec{r} \). We first check if \( \vec{F} \) is a gradient field. It is easy to find, by considering the equations \( f_x = 3x^2yz - 3y \), \( f_y = x^3z - 3x \), and \( f_z = x^3y + 2z \) that
\[ f(x, y, z) = x^3yz - 3xy + z^2 \]
satisfies \( \text{grad } f = \vec{F} \). Hence the fundamental theorem for line integrals implies that
\[ \text{work} = \int_C \vec{F} \cdot d\vec{r} = f(0, 3, 0) - f(0, 0, 2) = -4. \]
4. (10 points) Evaluate
\[ \oint_C xy \, dx + x^2y^3 \, dy \]
where $C$ is the positively oriented closed curve given in the picture.

**Solution:** We use Green’s theorem with $P = xy$ and $Q = x^2y^3$. Then
\[ \oint_C xy \, dx + x^2y^3 \, dy = \iint_D (Q_x - P_y) \, dA = \iint_D (2xy^3 - x) \, dA. \]
The integral can be computed by
\[ \iint_D (2xy^3 - x) \, dA = \int_0^1 \int_0^{2x} (2xy^3 - x) \, dy \, dx = \frac{2}{3}. \]
5. (10 points) (No partial points) (No need to explain) Each of the following four vector fields $\vec{F}(x, y, z)$ is shown in the $xy$-plane and looks the same in all other horizontal planes. In other words, $\vec{F}$ is independent of $z$ and its $z$-component is 0. Answer the following questions. You do not need to provide an explanation. No partial points are given for these questions.

(a) Find all points (among the 4 points A, B, C, D) at which $\text{div} \vec{F} \neq 0$.

The answer is A, C.

(b) Find all points (among the 4 points A, B, C, D) at which $\text{curl} \vec{F} \neq \vec{0}$.

The answer is B, D.

(c) Find all closed curves (among the 6 curves $C_1, C_2, \cdots, C_6$) along which the circulation $\oint_C \vec{F} \cdot d\vec{r}$ of the corresponding vector field is not zero.

The answer is $C_3, C_4, C_6$.

(d) For this question, regard the above vector fields as two-dimensional vector fields $\vec{G}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$. Find all closed curves (among the 6 curves $C_1, C_2, \cdots, C_6$) along which the flux $\oint_C \vec{G} \cdot \hat{n} \, ds$ of the corresponding vector field is not zero, where $\hat{n}$ is the outward unit normal vector.

The answer is $C_1, C_2, C_5$. 
6. (10 points) Consider the vector field

$$\vec{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$ 

This vector field satisfies \( \text{div} \vec{F}(x, y, z) = 0 \) for all \((x, y, z)\) except for \((0, 0, 0)\) at which the vector field is not defined. Let \( S \) be the closed surface given by the boundary of the box \(-2 \leq x \leq 2, -3 \leq y \leq 3, -4 \leq z \leq 4\). We orient \( S \) outward. Evaluate the flux of \( \vec{F} \) across \( S \).

**Solution:** This problem is basically same as Example 3 in Section 16.9. We use Divergence theorem for two surfaces. The answer is \(4\pi\).
7. (5 points) A particle starts at \((1, 1, 0)\) with initial velocity \(\langle 1, -1, 3 \rangle\). Its acceleration at time \(t\) is \(\langle 6t, 12t^2, -6t \rangle\). Where is the particle at time \(t\)?

**Solution:** Since \(\vec{a}(t) = \langle 6t, 12t^2, -6t \rangle\), \(\vec{v}(t) = \langle 3t^2, 4t^3, -3t^2 \rangle + \langle 1, -1, 3 \rangle\). Hence \(\vec{r}(t) = \langle t^3, t^4, -t^3 \rangle + \langle t, -t, 3t \rangle + \langle 1, 1, 0 \rangle\). Therefore, the particle is at \((1 + t + t^3, 1 - t + t^4, 3t - t^3)\) at time \(t\).

8. (5 points) Let \(f(x, y)\) be a function which has continuous partial derivatives. Consider the points \(A(1, 1), B(3, 1), C(0, 2),\) and \(D(4, 4)\). Suppose that the directional derivative of \(f\) at \(A\) in the direction of the vector \(\vec{AB}\) is 3 and the directional derivative of \(f\) at \(A\) in the direction of the vector \(\vec{AC}\) is \(\frac{1}{\sqrt{2}}\). Find the directional derivative of \(f\) at \(A\) in the direction of the vector \(\vec{AD}\).

**Solution:** From the definition of directional derivative, \(D_{\vec{AB}}f = \langle f_x, f_y, \rangle \cdot \frac{\vec{AB}}{|\vec{AB}|}\). We have \(\frac{\vec{AB}}{|\vec{AB}|} = \langle 1, 0 \rangle\), and hence \(D_{\vec{AB}}f = f_x\). Since we are given that \(D_{\vec{AB}}f = 3\), we find that \(f_x = 3\).

Similarly, we have \(\frac{\vec{AC}}{|\vec{AC}|} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle\), and the condition \(D_{\vec{AC}}f = \frac{1}{\sqrt{2}}\) implies that \(-\frac{f_x}{\sqrt{2}} + \frac{f_y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\).

Using \(f_x = 3\), we find that \(f_y = 4\).

Now, \(\frac{\vec{AD}}{|\vec{AD}|} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle\). Hence \(D_{\vec{AD}}f = \langle 3, 4 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{7}{\sqrt{2}}\).
9. (5 points) Suppose that the electric potential $V$ is given by $V(x, y, z) = 5x^2 - 3xy + xyz$. In which direction or directions does $V$ change most rapidly at $P(0, 1, 2)$?

**Solution:** $V$ change most rapidly in the directions of $\nabla V$ and $-\nabla V$. We have $\nabla V(0, 1, 2) = \langle -1, 0, 0 \rangle$. Hence $V$ changes most rapidly in the direction of the vectors $\langle -1, 0, 0 \rangle$ and $\langle 1, 0, 0 \rangle$.

10. (5 points) Find a parametric equation for the line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$.

**Solution:** Eliminate $x$ using the two equations and we obtain $3y + 2z = 0$. Hence if we set $z = t$, then $y = -\frac{2}{3}t$. Using the equation $x - y + z = 1$, we find that $x = 1 - \frac{5}{3}t$. Hence the parametric equation of the line is $\vec{r}(t) = \langle 1 - \frac{5}{3}t, -\frac{2}{3}t, t \rangle$. 