1. (10 point) Let \( f(x, y, z) \) be a function which has continuous partial derivatives. Suppose that the directional derivative of \( f \) at \( P(1, 2, 3) \) in the direction of \( \vec{a} = \vec{i} + \vec{j} \) is 4, the directional derivative of \( f \) at \( P(1, 2, 3) \) in the direction of \( \vec{b} = \vec{j} + \vec{k} \) is 3, and the directional derivative of \( f \) at \( P(1, 2, 3) \) in the direction of \( \vec{c} = \vec{i} + \vec{k} \) is 1. Find the gradient of \( f \) at \( P(1, 2, 3) \).

At \( P \), \( 4 = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}}{\sqrt{2}} \cdot (f_x, f_y, f_z) \cdot \frac{(1, 1, 0)}{\sqrt{2}} \)

\[ \Rightarrow f_x + f_y = 4 \sqrt{2} \]

Similarly, \( f_y + f_z = 3 \sqrt{2} \), \( f_x + f_z = \sqrt{2} \)

Solving the equations, \( f_x = \sqrt{2}, \; f_y = 3\sqrt{2}, \; f_z = 0 \)

The gradient of \( f \) at \( P(1, 2, 3) \) is \( \langle \sqrt{2}, 3\sqrt{2}, 0 \rangle \).
2. (5 point) A particle is moving in space. It starts off at the origin with initial velocity $\vec{v} = \vec{i} - \vec{j} + 3\vec{k}$. Due to the gravity and its own propulsion, the particle’s acceleration is $\vec{a}(t) = 6t \vec{i} + 12t^2 \vec{j} - 6t \vec{k}$ as a function of time $t$. Find the position of the particle at time $t$.

\[
\vec{a}(t) = \langle 6t, 12t^2, -6t \rangle
\]

\[
\vec{v}(t) = \langle 3t^2, 4t^3, -3t^2 \rangle + \vec{v}(0)
\]

\[
\vec{v}(t) = \langle 3t^2 + 1, 4t^3 - 1, -3t^2 + 3 \rangle
\]

\[
\vec{r}(t) = \langle t^3 + t, t^4 - t, -t^2 + 3t \rangle + \vec{r}(0)
\]

But $\vec{r}(0) = \langle 0, 0, 0 \rangle$

The position of the particle at time $t$ is $\langle t^3 + t, t^4 - t, -t^2 + 3t \rangle$.
3. (10 points) Let $C$ be the curve given in the picture. The curved part of $C$ is a portion of a circle centered at $(0, 0)$. Evaluate the work done by the vector field

$$\vec{F}(x, y) = (x^2y, -xy^2)$$

in moving a particle along $C$.

\[
\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \int_C x^2y \, dx - xy^2 \, dy
\]

\[
= \iint_D (Ax - Py) \, dA = \iint_D (-y^2 - x^2) \, dA
\]

\[
= \iint_D \frac{5\pi}{4} r \, dr \, d\theta
\]

\[
= \left[ -\frac{3}{2} \right] \pi
\]

The work is $-\frac{3}{2} \pi$. 

\[
-\frac{3}{2} \pi
\]
4. (10 point) Let
\[ \vec{F}(x, y, z) = (x^2, 2xy + \sin z, z). \]
Evaluate the flux of the vector field \( \vec{F} \) across the boundary surface of the unit cube, given with positive (outward) orientation, in the picture.

\[ \text{Flux} = \iiint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div} \vec{F} \, dV \]

\[ = \iiint_E (2x + 2x + 1) \, dV \]

\[ = \iiint_E (4x + 1) \, dz \, dy \, dx = 3 \]

The flux is 3.
5. (10 point) Let $C$ be the part of the curve $x + y^2 = 4$ from point $(-5, 3)$ to point $(0, 2)$. Evaluate the integral

$$\int_C y^2 \, dx - 2xy \, dy.$$

Parametrize the curve by $y = t$, $x = 4 - t^2$,

$$-3 \leq t \leq 2$$

$$\Rightarrow \int_C y^2 \, dx - 2xy \, dy$$

$$= \int_{-3}^{2} t^2 (-2t) \, dt - 2(4 - t^2)t \, dt$$

$$= \int_{-3}^{2} (-2t^3) \, dt = 20$$

The integral is 20.
6. (10 points; No partial points) Consider the following vector fields \( \vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j} \).
Answer the following questions. You do not need to provide an explanation.

![Diagram](image)

(i) Is \( \nabla \cdot \vec{F} \) at point A in picture (a) 0, positive, or negative? It is _____

(ii) Is \( \nabla \cdot \vec{F} \) at point B in picture (b) 0, positive, or negative? It is _____

(iii) Is \( \nabla \cdot \vec{F} \) at point C in picture (b) 0, positive, or negative? It is _____

For the next two questions, consider the three-dimensional vector fields \( \vec{G}(x, y, z) = P(x, y)\vec{i} + Q(x, y)\vec{j} \) made out of the vector field \( P(x, y)\vec{i} + Q(x, y)\vec{j} \) in Picture (c) and (d). In other words, \( \vec{G}(x, y, z) \) is given as Picture (c) and (d) on every plane parallel to the \( xy \)-plane.

(iv) Is \( (\nabla \times \vec{G}) \cdot \vec{k} \) at point D in picture (c) 0, positive, or negative? It is _____

(v) Is \( (\nabla \times \vec{G}) \cdot \vec{k} \) at point E in picture (d) 0, positive, or negative? It is _____
7. (10 point) Consider the vector field

\[ \vec{F}(x, y, z) = (z^2 + y \sin(yz), 2xze^{x^2} - y - z, x^2 + y^2 + z). \]

It is known that \( \vec{F} = \text{curl} \vec{G} \) for some vector field \( \vec{G}(x, y, z) \). (You do not need to check this fact.) Now consider the sphere \( x^2 + y^2 + z^2 = 1 \). The plane \( z = -\frac{1}{2} \) divides the sphere into two parts. Let \( S \) be the smaller part that is below the plane. Evaluate the integral

\[ \iint_S \vec{F} \cdot d\vec{S} \]

where \( S \) is oriented "downward" (i.e. the \( k \)-component of the normal vector \( \vec{n} \) to \( S \) is negative).

By Stokes' theorem, twice,

\[ \iint_S \vec{F} \cdot d\vec{S} = \iint_S \text{curl} \vec{G} \cdot d\vec{S} \]

\[ = \oint_C \vec{G} \cdot d\vec{r} = \iint_D \text{curl} \vec{G} \cdot d\vec{S} \]

where \( D \) is the disc \( x^2 + y^2 = \frac{3}{4}, z = -\frac{1}{2} \), with normal vector pointing downward. Thus, \( \iint_D \vec{F} \cdot d\vec{S} = \iint_D -((x^2 + y^2 + z)) \, dA \)

\[ \vec{n} = \langle 0, 0, -1 \rangle \]

The integral is

\[ \frac{3}{32} \pi \]
8. (10 point) Evaluate the integral

\[ \iint_S \langle x, y, 1 \rangle \cdot d\mathbf{S} \]

where \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies above the cone \( z = \frac{x^2 + y^2}{2} \). Here the surface \( S \) is oriented “upward” (i.e. the \( \mathbf{k} \)-component of the normal vector \( \mathbf{n} \) to \( S \) is positive.)

On sphere,

\[ dS = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} \quad dS \]

\[ = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} \quad dS \]

\[ x^2 + y^2 + z^2 = 4 \quad \text{on sphere} \]

\[ \iint_S \langle x, y, 1 \rangle \cdot d\mathbf{S} = \iint_S \frac{x^2 + y^2 + z}{2} \quad dS = \iint_S \frac{4 - z^2 + z}{2} \quad dS \]

\[ = \int_0^{2\pi} \int_0^{\pi/4} \frac{4 - (2 \cos \phi)^2 + 2 \cos \phi (2 \sin \phi)}{2} \quad d\phi \quad d\theta \]

\[ = 2\pi \left( \frac{\frac{19}{3}}{3} - \frac{10}{3} \sqrt{2} \right) \]

The integral is

\[ 2\pi \left( \frac{\frac{19}{3}}{3} - \frac{10}{3} \sqrt{2} \right) \]
9. (10 point) Consider the vector field

\[ \vec{F}(x, y, z) = (y^2, 2xy + e^{3z}, 3ye^{3z}). \]

Evaluate the work done by \( \vec{F} \) in moving a particle from point \((0, 0, 0)\) to point \((1, 1, 1)\) along each of the following paths.

(a) \( C_1 \) is the straight line segment from \((0, 0, 0)\) to \((1, 1, 1)\).
(b) \( C_2 \) consists of three line segments, the first from \((0, 0, 0)\) to \((1, 0, 0)\), the second from \((1, 0, 0)\) to \((1, 1, 0)\), and the third from \((1, 1, 0)\) to \((1, 1, 1)\).
(c) \( C_3 \) is the curve \((t, t^2, t^3), 0 \leq t \leq 1\).
(d) \( C_4 \) is the curve \((t \sin(\frac{\pi}{2}t^2), te^{t^2-1}, t^3), 0 \leq t \leq 1\).

Observe that \( \vec{F} \) is a conservative vector field. This can be seen either by noting that

\[ \vec{F} = \nabla f \] where \( f(x, y, z) = xy^2 + ye^{3z} \)

or by checking that \( \nabla \times \vec{F} = \vec{0} \) and \( \vec{F} \) is defined in all of \( \mathbb{R}^3 \).

Thus, the integral \( \int_{C} \vec{F} \cdot d\vec{r} = f(\text{end point}) - f(\text{initial point}) \)

\[ = f(1,1,1) - f(0,0,0) \] in all 4 cases

\[ = (1+e^3) - 0 = 1+e^3 \]

(a) The work along \( C_1 \) is \( 1+e^3 \)
(b) The work along \( C_2 \) is \( 1+e^2 \)
(c) The work along \( C_3 \) is \( 1+e^3 \)
(d) The work along \( C_4 \) is \( 1+e^3 \)