1. Consider the double integral
\[ \int \int_D \frac{dA}{\sqrt{x^2 + y^2}}. \]

(a) (7 points) Evaluate the double integral if \( D \) is the unit disc \( x^2 + y^2 \leq 1 \).

(b) (3 points) Suppose now that \( D \) is the solid square with vertices at \((\pm 1, \pm 1)\). Is this double integral greater, equal to, or less than the answer to part (a)?

**Answer:** (a) \(2\pi\). (b) Greater.

2. This problem has two parts.

(a) (6 points) Find the area of the triangle with vertices at \((0, 0, 0), (1, 1, 4), \) and \((-2, 1, -2)\).

(b) (4 points) Suppose \( a = i + j + 4k \) and \( b = -2i + j - 2k \). If \( c = 2i + j + 3k \), find the vector component of \( c \) that is perpendicular to the plane defined by \( a \) and \( b \). All vectors are assumed to originate at the origin.

**Answer:** (a) The area vector of the parallelogram is \((-6, -6, 3)\) and the area of the triangle is \(9/2\). (b) \(\frac{1}{3}(2i + 2j - k)\).

3. This problem has three parts.

(a) (2 points) Find the distance of the origin from the plane \(2x + 3y - 6z = 14\).

(b) (3 points) Find the point on the plane \(2x + 3y - 6z = 14\) that is closest to the origin.

(c) (5 points) The lines \((x, y, z) = (2, 2t + 1, t + 1)\) and \((x, y, z) = (3t - 2, 3, t + 1)\) do not intersect. Find the distance between the two lines.

**Answer:** (a) 2. (b) \(\frac{2}{7}(2, 3, -6)\). (c) The vector \(2i + 3j - 6k\) is orthogonal to both lines. The unit vector in that direction is \(u = \frac{1}{7}(2i + 3j - 6k)\). The distance is

\[ ((2, 1, 1) - (-2, 3, 1)) \cdot u \]

or \(\frac{2}{7}\).

4. This problem has three parts

(a) (2 points) Find \(\frac{\partial u}{\partial r}\) if \(u = x^2 + y\) and \(x = r \cos \theta, \ y = r \sin \theta\).

(b) (4 points) Find \(\frac{\partial u}{\partial r}\) if \(u = f(x, y)\) and \(x = r \cos \theta, \ y = r \sin \theta\).

(c) (4 points) Find \(\frac{\partial u}{\partial x}\) if \(u = g(r, \theta)\) and \(x = r \cos \theta, \ y = r \sin \theta\).

**Answer:** (a) \(2x \cos \theta + \sin \theta\). (b) \(\frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta\). (c) Differentiate \(x = r \cos \theta\) as well as \(y = r \sin \theta\) with respect to \(x\) to get

\[
1 = \cos \theta \frac{\partial r}{\partial x} - r \sin \theta \frac{\partial \theta}{\partial x},
\]

\[
0 = \sin \theta \frac{\partial r}{\partial x} + r \cos \theta \frac{\partial \theta}{\partial x}.
\]

Therefore \(\frac{\partial r}{\partial x} = \cos \theta\) and \(\frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}\). It follows that

\[
\frac{\partial u}{\partial x} = \frac{\partial g}{\partial r} \cos \theta - \frac{\partial g}{\partial \theta} \frac{\sin \theta}{r}.
\]
5. Both parts ask you to reverse the order of integration.

(a) (4 points) Rewrite the integral
\[ \int_{x=0}^{1} \int_{y=0}^{x/2} f(x, y) \, dy \, dx \]
with x inner and y outer.

(b) (6 points) Rewrite the integral
\[ \int_{x=0}^{1} \int_{y=x^2}^{1} f(x, y) \, dy \, dx \]
with x inner and y outer.

Answer: (a) \( \int_{y=0}^{1/2} \int_{x=0}^{2y} \) (b) \( \int_{x=0}^{1} \int_{y=x^2}^{1} \).

6. Consider the helix \( (x, y, z) = (\cos t, \sin t, t) \), with t being the parameter.

(a) (2 points) If t is time and \( (\cos t, \sin t, t) \) is the position of a particle at time t, find the magnitude of its acceleration.

(b) (3 points) Find the length of the helix from \( t = 0 \) to \( t = 2\pi \).

(c) (5 points) Assume that the density (mass per unit length) of the helix from \( t = 0 \) to \( t = 2\pi \) is constant and equal to 1. Find \( \bar{z} \), the z-coordinate of the center of mass of the part of the helix from \( t = 0 \) to \( t = 2\pi \).

Answer: (a) 1. (b) \( 2\sqrt{2}\pi \). (c) \( \pi \).

7. In each part, the curve C is assumed to be counterclockwise. Evaluate
\[ \int_{C} y \, dx \]
for the following C:

(a) (2 points) C is the circle \( x^2 + y^2 = 4 \).

(b) (4 points) C is the square with vertices at \( (\pm 1, \pm 1) \).

(c) (4 points) C is the curve below (the arcs are semicircles):

Answer: (a) \(-4\pi \). (b) \(-4 \). (c) \(-4 - 2\pi \).

8. Let \( S \) be the hemispherical surface \( x^2 + y^2 + z^2 = 1 \) between the planes \( z = 0 \) and \( z = 1 \). The normal to the surface or \( dS \) is assumed to be pointing out of the center of the hemisphere.
(a) (5 points) Find the flux
\[ \int \int_S F \cdot dS \]
with \( F = zk \).
(b) (5 points) Find the flux
\[ \int \int_S \text{curl} F \cdot dS \]
with \( F = -yi + xj + zk \).

**Answer:** (a) \( \frac{2\pi}{3} \). (b) \( 2\pi \).

9. The position vector is given by \( r = xi + yj + zk \).

(a) (3 points) Let \( F = |r|^2 r \). Find \( \text{div} F \).
(b) (3 points) Again let \( F = |r|^2 r \). Find the outward flux
\[ \int \int_S F \cdot dS \]
with \( S \) being the surface of the cube with vertices at \((\pm1, \pm1, \pm1)\).
(c) (4 points) Now suppose \( F = \frac{r}{|r|^3} \). Find the outward flux
\[ \int \int_S F \cdot dS \]
with \( S \) being the surface of the cube with vertices at \((\pm1, \pm1, \pm1)\).

**Answer:** (a) \( 5(x^2 + y^2 + z^2) \). (b) \( 40 \). (c) The divergence of \( F \) is zero everywhere except at the origin, where it generates a flux of \( 4\pi \). Because the origin is inside the cube, the flux out of the cube is \( 4\pi \).