Math 215 — Final Exam
December 17, 2019

First 3 Letters of Last Name: □ □ □  UM Id#: __________________
Instructor: ___________________________  Section: __________________

1. **Do not open this exam until you are told to do so.**

2. This exam has 17 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.

3. Do not separate the pages of this exam, other than the formula sheet at the end of the exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.

4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.

5. The true or false questions are the only questions that do not require you to show your work. For all other questions show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.

6. You may use no aids (e.g., calculators or notecards) on this exam.

7. **Turn off all cell phones**, remove all headphones, and **place any watch you are using on the desk in front of you**.

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1. [10 points] The graph below is a plot of some of the level curves of a function $g$ in a rectangular region $R = [4, 4] \times [4, 4]$. Assume that as we move between adjacent level curves the value of $g$ increases or decreases by exactly one. The arrows point in the direction of $\nabla g$.

![Graph of level curves with arrows indicating the direction of \( \nabla g \).]

a. [4 points] Suppose $C$ is the line segment with initial point at $(0.9, 3.2)$ and final point at $(3.5, 1.0)$. What is the approximate value of $\int_C \nabla g \cdot dr$? 

\footnote{within .2}
b. [6 points] Identify the approximate coordinates\(^2\) of the critical points of \(g\) in \(R\). For each critical point, indicate if it is a point where the function has a local maximum, a local minimum, or a saddle.

\(^2\)That is, within .1 in each coordinate direction.
2. [12 points] Some calculations.
   \(\text{a. [4 points]}\) Compute \(\text{curl}(\mathbf{F})\) for \(\mathbf{F} = (xyz, y\sin(z), y\cos(x))\).
b. [8 points] Suppose \( g(x, y, z) = e^{2y}(x^2 + z^3) \). If \( x = uvw, y = vw/u \) and \( z = 2u + vu + w \), then find \( g_w \), the partial derivative of \( g \) with respect to \( w \), when \( u = -1, v = 2, \) and \( w = 1 \).
3. [8 points] As usual, justify your answers.

a. [4 points] Suppose that $D$ is the bounded region in the plane that has boundary given by the oriented simple closed piece-wise smooth curves $C_1$, $C_2$, $C_3$, and $C_4$ as in the picture. Suppose $\mathbf{F} = \langle P, Q, 0 \rangle : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field and $P$ and $Q$ have continuous partial derivatives on $\mathbb{R}^2$. If

$$\oint_{C_k} \mathbf{F} \cdot d\mathbf{r} = 2^k,$$

find $\iint_D (Q_x - P_y) \, dA = \iint_D (\nabla \times \mathbf{F}) \cdot \hat{k} \, dA$. 

The diagram shows a region $D$ with boundaries $C_1$, $C_2$, $C_3$, and $C_4$. The orientation of the curves is indicated by the arrows.
b. [4 points] Suppose $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth vector field. Let $\tilde{Q}$ be the solid cube centered at the origin with vertices $(\pm 2, \pm 2, \pm 2)$, let $\tilde{S}$ be the sphere centered at the origin of radius 10, and let $E$ be the bounded solid region between $\tilde{S}$ and $\partial \tilde{Q}$. Suppose that the outward flux of $\mathbf{F}$ across $\tilde{S}$ is $5\pi$ and the divergence of $\mathbf{F}$ on $\tilde{Q}$ is $\pi/16$. If possible, evaluate the following integral
\[ \iiint_E (\nabla \cdot \mathbf{F}) \, dV. \]
4. [10 points] Suppose $R$ is a positive real number. Let $S$ be the cone given by the equation $z = \sqrt{x^2 + y^2}$ with $0 \leq z \leq R$, oriented downward. Compute the flux of $\mathbf{G} = \langle xz, yz, xy \rangle$ across $S$. 
5. [8 points] Identify the graph which best represents the vector fields below.

- \( F(x,y) = (y, 0) \)
- \( G(x,y) = (x, \cos(y)) \)
- \( H(x,y) = (\sin(y), x) \)
- \( I(x,y) = (x, -y) \)
6. [14 points] A failure to justify your answer(s) could result in zero points. The torus $T$, pictured below, is oriented outward and parameterized by

$$r(u, v) = ((3 + \cos(u)) \cos(v), (3 + \cos(u)) \sin(v), \sin(u))$$

for $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$.

a. [2 points] Does $(0, 3, 1)$ belong to the surface? If so, what are the values of $u$ and $v$ so that $r(u, v) = (0, 3, 1)$?

b. [8 points] Find an equation for the tangent plane to $T$ at $r(\pi/3, \pi/6) = (7\sqrt{3}/4, 7/4, \sqrt{3}/2)$. 
c. [4 points] Compute \[ \iint_T (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \]
for \( \mathbf{F} = \langle xyz, y \sin(z), y \cos(x) \rangle \).
7. [10 points] Let \( \alpha : \mathbb{R}^3 \to \mathbb{R} \) be a continuous function. Consider the integral

\[
\int_0^3 \int_0^{4-(4/3)x} \int_0^{5-(5/3)x-(5/4)y} \alpha(x, y, z) \, dz \, dy \, dx.
\]

a. [4 points] Sketch the region of integration.
b. [6 points] If we change the order of integration to
\[
\int_a^b \int_c^d \int_e^f \alpha(x, y, z) \, dx \, dz \, dy,
\]
then
\begin{itemize}
  \item \( a \) is ______________________
  \item \( b \) is ______________________
  \item \( c \) is ______________________
  \item \( d \) is ______________________
  \item \( e \) is ______________________
  \item \( f \) is ______________________
\end{itemize}
8. [10 points] Suppose $C$ is a smooth simple closed curve in the plane and $A = (a_1, a_2)$ is a point in the plane outside of the region bounded by the curve $C$. Let $B = (b_1, b_2)$ be a point on the curve closest to $A$.

a. [2 points] Draw a picture that illustrates the situation.

b. [8 points] Use the method of Lagrange multipliers to explain why the tangent line to $C$ at the point $B$ is perpendicular to the line that contains $A$ and $B$. 
9. [8 points] The following equations are expressed in spherical coordinates. Sketch the solid or surface described by each equation.
   a. [4 points] $8 - \rho \sin \phi = 0$
   b. [4 points] $\rho^2 = 6\rho - 8$
10. [10 points] Indicate if each of the following is true or false by circling the correct answer. No partial credit will be given.

a. [2 points] Consider the function $y = 4\cos(t)$ where $0 \leq t \leq 5$ is measured in seconds. The following commands are entered in MatLab:

$\gg t = 0:.01:5$;
$\gg y = 4*\cos(t)$;

If you type $y(1)$, then MatLab will return 4.

True  False

b. [2 points] If $G: \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth vector field, then $\nabla(\text{div } G) = 0$.

True  False

c. [2 points] The result of integrating a vector field over a surface is a vector.

True  False

d. [2 points] The integral of a vector field over a smooth closed curve is always zero.

True  False

e. [2 points] If $x = uv$ and $y = u^2 - v^2$, then the Jacobian of the transformation is $2u^2 + 2v^2$.

True  False
\[ \sin^2(x) + \cos^2(x) = 1, \quad \cos(2x) = \cos^2(x) - \sin^2(x), \quad \sin(2x) = 2 \sin(x) \cos(x) \]
\[ \sin^2(x) = \frac{1 - \cos(2x)}{2}, \quad \cos^2(x) = \frac{1 + \cos(2x)}{2} \]
\[ \cos(\pi/3) = \frac{1}{2}, \quad \sin(\pi/3) = \frac{\sqrt{3}}{2}, \quad \cos(\pi/4) = \frac{\sqrt{2}}{2}, \quad \sin(\pi/4) = \frac{\sqrt{2}}{2}, \quad \cos(\pi/6) = \frac{\sqrt{3}}{2}, \quad \sin(\pi/6) = \frac{1}{2}, \quad \cos(0) = 1, \quad \sin(0) = 0. \]
\[ \frac{d}{dx} \sin(x) = \cos(x), \quad \frac{d}{dx} \cos(x) = -\sin(x). \]

Volume of the parallelepiped determined by the vectors \( \mathbf{v}_1 = \langle a, b, c \rangle, \mathbf{v}_2 = \langle d, e, f \rangle, \) and \( \mathbf{v}_3 = \langle g, h, i \rangle \) is \[ |\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)| = \text{absolute value of} \[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{vmatrix}
\]

Distance from a point \((a, b, c)\) to a plane \(Ax + By + Cz + D = 0\) is \[ \frac{|Aa + Bb + Cc + D|}{\sqrt{A^2 + B^2 + C^2}}. \]

The circumference of a circle of radius \(a\) is \(2\pi a\).

The area of a disk of radius \(a\) is \(\pi a^2\).

The volume of a right circular cylinder of radius \(a\) and height \(h\) is \(\pi a^2 h\).

The curvature of the curve given by the parametric equation \(r(t)\) is \(\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}\).

\[ \int \sin^2(u) \, du = \frac{u}{2} - \frac{\sin(2u)}{4} + C \]
\[ \int \cos^2(u) \, du = \frac{u}{2} + \frac{\sin(2u)}{4} + C \]
\[ \int \ln(u) \, du = u \ln(u) - u + C \]

The volume of a right circular cylinder of radius \(a\) and height \(h\) is \(\pi a^2 h\).

The volume of a sphere of radius \(a\) is \(\frac{4}{3}\pi a^3\).

The surface area of a sphere of radius \(a\) is \(4\pi a^2\).

The volume of a cone with base radius \(a\) and height \(b\) is \(\frac{1}{3} \pi a^2 b\).

Polar coordinates \(x = r \cos(\theta), y = r \sin(\theta)\).

Cylindrical coordinates \(x = r \cos(\theta), y = r \sin(\theta), z = z\).

Spherical coordinates \(x = \rho \cos(\theta) \sin(\phi), y = \rho \sin(\theta) \sin(\phi), z = \rho \cos(\phi)\).

Green’s Theorem:
\[ \oint_D P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA. \]

Stokes’ Theorem:
\[ \oint_S \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} \]

Divergence Theorem:
\[ \iiint_E \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\text{div} \mathbf{F}) \, dV. \]