

Name: _____

Lab Section: _____

MATH 215 – Winter 2005
FINAL EXAM

Show your work in this booklet.
Do NOT submit loose sheets of paper—They won't be graded

Problem	Points	Score
1	10	
2	15	
3	20	
4	10	
5	10	
6	15	
7	10	
8	10	
TOTAL	100	

Some useful trigonometric identities:

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

Spherical coordinates:

$$x = \rho \cos(\theta) \sin(\phi)$$

$$y = \rho \sin(\theta) \sin(\phi)$$

$$z = \rho \cos(\phi)$$

Problem 1. (5+5=10 points) In this problem $f(x, y) = y^2x - x^2 + 2xy$ and P is the point $P = (2, 1)$.

(a) In what direction is the rate of change of f greatest at P ? Express your answer in terms of a **unit** vector.

(b) Suppose $\vec{r}(t) = \langle x(t), y(t) \rangle$ is a parametric curve such that $\vec{r}(0) = \langle 2, 1 \rangle$, and $\frac{d}{dt}\vec{r}(0) = \langle 3, 5 \rangle$. Find the value of

$$\frac{d}{dt}f(x(t), y(t))|_{t=0}.$$

Problem 2. (8+7=15 points) Suppose that, in an experiment, the temperature of a sample (in degrees Celcius) is given by the function $T(x, y, z) = 2y^2 + ze^{-x} + 16$, where x , y and z are variables one can control in the experiment.

(a) Using the linear approximation of the function T at the point $(x_0, y_0, z_0) = (0, 1, 2)$, find an approximate value of $T(0.2, 0.9, 2.3)$. Note that $T(0, 1, 2) = 20^\circ$,

(b) Suppose one wants to change (x_0, y_0, z_0) a little and yet maintain the temperature at 20° . Using the linear approximation, **find an equation** between Δx , Δy and Δz so that

$$T(\Delta x, 1 + \Delta y, 2 + \Delta z) \cong 20.$$

Problem 3. ($4 \times 5 = 20$ points) For each item, circle the correct answer or indicate if the statement is true or false. Assume that the functions, fields and curves below are smooth.

Think carefully before you answer –no partial credit on this one, -take your time!

(a) Let \mathcal{C} be an arc from $(0, 0)$ to $(2, 1)$. According to the fundamental theorem for line integrals, $\int_{\mathcal{C}} (y - 1) dx + (x + 2y) dy$ is equal to:

(1) 2

(2) 1

(3) It depends on what \mathcal{C} is.

(b) For every smooth function f , the integral $\int_0^1 \int_0^{2y^2+1} f(x, y) dx dy$ is equal to

(1) $\int_0^3 \int_0^{\frac{1}{2}\sqrt{x-1}} f(x, y) dy dx$

(2) $\int_1^3 \int_0^{\frac{1}{2}\sqrt{x+1}} f(x, y) dy dx$

(3) None of the above.

(c) If (a, b) is a critical point of a function f , and if

$$f_{xx}(a, b) = -2 \quad \text{and} \quad f_{yy}(a, b) = 3,$$

then what can one say about (a, b) ?

(1) Nothing can be concluded from the given information.

(2) (a, b) is a local minimum of f .

(3) (a, b) is a local maximum of f .

(4) (a, b) is a saddle point of f .

(d) If \vec{F} is a field such that $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$ where \mathcal{C} is the unit circle, then \vec{F} must be conservative.

(1) True.

(2) False.

(e) If \mathcal{C} is the boundary of a domain D and \mathcal{C} is oriented as in the statement of Green's theorem, then $\oint_{\mathcal{C}} x^2y dx - y dy$ equals

(1) $\iint_D (2xy - 1) dA$.

(2) $\iint_D (1 - x^2) dA$.

(3) $\iint_D (-x^2) dA$.

(4) None of the above.

Problem 4. (10 points) Let $a > 0$ denote a fixed constant. A uniform plate with mass density 2 gr/cm^2 occupies the region bounded by the curves:

$$y = \sqrt{x} \quad \text{with } 0 \leq x \leq a, \quad y = \sqrt{-x} \quad \text{with } -a \leq x \leq 0, \quad \text{and } y = \sqrt{a}$$

Find the coordinates of the center of mass of the plate.

Problem 5. (5+5=10 points) Let \mathcal{C} denote the oriented closed curve consisting of the line segment from $(0, 0)$ to $(\sqrt{2}, 0)$, followed by the arc of the circle $x^2 + y^2 = 2$ from $(\sqrt{2}, 0)$ to $(1, 1)$, followed by the line segment from $(1, 1)$ to $(0, 0)$.

(a) By an explicit direct calculation, compute $I = \oint_{\mathcal{C}} y \, dx$. (You have to break the calculation into three line integrals.)

(b) Verify your answer to part (a) by computing I using Green's theorem.

Problem 6. (7+8=15 points) Let S be the portion of the cylinder given in cylindrical coordinates by

$$0 \leq z \leq 3, \quad r = 1, \quad 0 \leq \theta \leq \pi/2.$$

Orient S by normal vectors pointing away from the z axis.

(a) Compute the flux (surface integral) of $\vec{F} = \langle 2x, y, -3z \rangle$ across S .

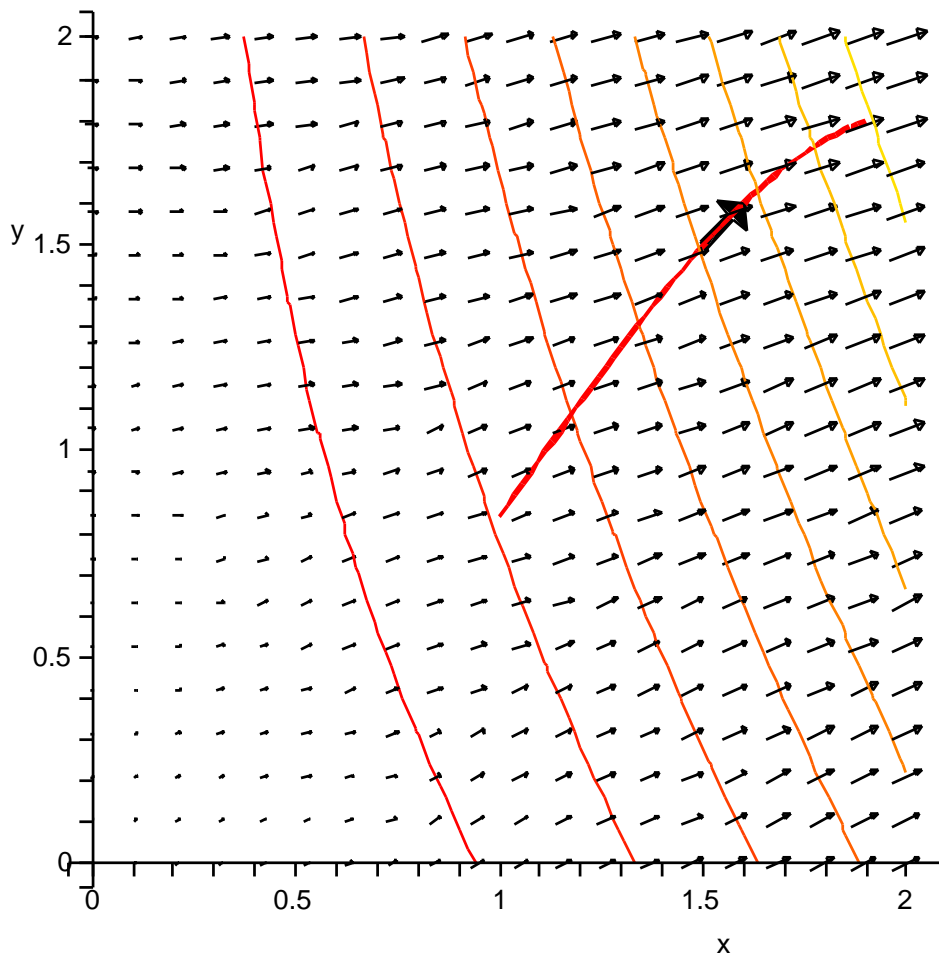
(b) Let \mathcal{C} denote the boundary of S , oriented counter-clockwise if one looks at S from the point $(5, 5, 1)$. Consider the line integral $I = \oint_{\mathcal{C}} yzdx - 2xzdy$.

Without computing the numerical value of I , determine whether I equals the surface integral of part (a). Justify your conclusion carefully.

Problem 7. (10 points) Let E denote the portion of the solid sphere of radius R in the first octant, and let

$$\vec{F} = (2x + y)\vec{i} + y^2\vec{j} + \cos(xy)\vec{k}$$

Applying the Divergence Theorem, compute the net flux of the field (surface integral) \vec{F} across the boundary of E , oriented by the outward-pointing normal vectors.



Problem 8. (5+5=10 points) The figure above is a contour plot of a function f and of its gradient. The values of f on two adjacent level curves differ by 5 units.

The plot also includes an oriented curve, \mathcal{C} .

(a) What is a pretty good estimate of the value of the line integral $\int_{\mathcal{C}} \nabla f \cdot d\vec{r}$?

- (1) 30
- (2) -29
- (3) The integral cannot be estimated with the given data.

(b) According to the plot, which of the following appears to hold?

- (1) $\operatorname{div}(\nabla f)(1.7, 1) > 0$.
- (2) $\operatorname{div}(\nabla f)(1.7, 1) < 0$.
- (3) $\operatorname{div}(\nabla f)(1.7, 1) = 0$ because $\nabla \cdot \nabla f = 0$ for all smooth functions f .