

MATH 216 Introduction to Differential Equations

Fall 2007 First Midterm Exam

Name _____

Section _____

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- This examination booklet contains 8 problems on 9 pages.
 - This is a closed book exam. No calculators are allowed. One 3 by 5 note card is allowed.
 - **Show all your work. No work, no points!**

Problem	Possible score	Your score
1	20	
2	13	
3	10	
4	10	
5	10	
6	12	
7	13	
8	12	
Total	100	

1. (10 Points) Solve the equation

$$\frac{dy}{dx} = \frac{\sqrt{y}}{1+x^2}, \quad y(1) = 1$$

Solution:

$$\frac{dy}{\sqrt{y}} = \frac{dx}{1+x^2} \quad 2 \text{ pts}$$

$$2\sqrt{y} = \arctan x + c \quad 3 \text{ pts}$$

$$2 = \frac{\pi}{4} + c \quad 2 \text{ pts}$$

$$2\sqrt{y} = \arctan x + 2 - \frac{\pi}{4}$$

$$y = 1/2(\arctan x + 2 - \frac{\pi}{4})^2 \quad 3 \text{ pts}$$

2. (10 Points) Consider

$$x^2 \frac{dy}{dx} + xy = \frac{1}{x}, y(1) = 1$$

(a) Determine the integrating factor $\rho(x)$.

$$\rho = x$$

2 pts

(b) Rewrite the differential equation so that the left hand side is a perfect derivative.

$$\frac{d(xy)}{dx} = \frac{1}{x^2}$$

2 pts

(c) Solve for $y(x)$.

Solution:

$$y = \frac{1}{x} \left[\int \frac{1}{x^2} + C \right]$$

$$y = -1/x^2 + C/x$$

$$y = -1/x^2 + 2/x$$

6 pts

3. (15 Points) Consider the following initial value problem

$$\begin{cases} \frac{dx}{dt} = 10x - 5x^2 \\ x(0) = C \end{cases}$$

(a) (4 pts) Find the equilibrium solutions and determine whether they are stable or not.

The equilibrium solutions are $x = 0$ (unstable) and $x = 2$ (stable). 4 pts

(b) (9 pts) Find the general and particular solution. Are there any singular solutions?

By separation of variables and using partial fractions, we get

$$\int \frac{1}{x(2-x)} dx = \int 5 dt$$

$$\frac{1}{2} \int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{2-x} dx = \frac{1}{2} \ln \left| \frac{x}{2-x} \right|$$

$$x(t) = \frac{2}{1 + Ke^{-10t}} \quad K \in \mathbb{R}$$

by plugging in the initial condition, we get

$$K = \frac{2}{C} - 1$$

hence, we have the particular solution

$$x(t) = \frac{2}{1 + \left(\frac{2}{C} - 1\right)e^{-10t}}$$

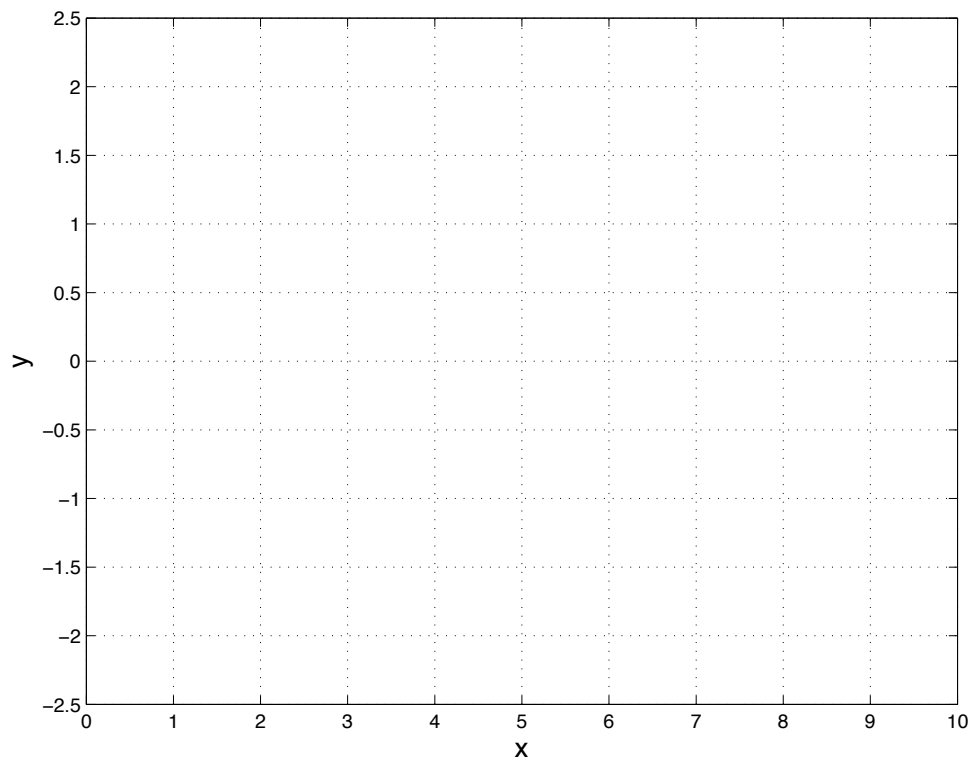
Singular solutions are $x = 0$, $x = 2$. 9 pts

(c) (2 pts) Compute $\lim_{t \rightarrow -\infty} x(t)$ if $0 < C < 2$.

If $0 < C < 2$, $\lim_{s \rightarrow -\infty} x(t) = 0$, since $0 < x(t) < 2$ and x is increasing. 2 pts

4. (10 Points) Draw the slope fields and typical solution curves for the equation

$$\frac{dy}{dx} = 1 - y^2.$$



5. (10 Points) Find the general solution of the differential equation:

$$y'' + 4y' + 20y = 0.$$

Aux. equation is $m^2 + 4m + 20 = 0$ which has solutions $m = -2 \pm 4i$, therefore $y = e^{-2x} (c_1 \cos(4x) + c_2 \sin(4x))$.

6. (10 Points)

(a) (5 pts) Find the square root of $-5 + 12i$.
 $\pm(3 + 3i)$.

(b) (5 pts) Solve $z^2 - 2z + 2 = 0$ and express the solution as $x + iy$.
 $z = 1 \pm i$

7. (15 Points) Leaves in a forest fall and accumulate on the ground at a rate of r kg per year. At the same time, the fallen leaves decompose at a rate proportional to the leaf mass on the ground, with proportionality factor α . Assume that the constants $r, \alpha > 0$.

(a) (2 pts) Let $m(t)$ be the leaf mass on the ground. Give the differential equation for $m(t)$.

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$$\frac{dm}{dt} = r - \alpha m \quad (2 \text{ pts})$$

(b) (3 pts) Over a long period of time, the leaf mass approaches an equilibrium value M . Find M in terms of the annual accumulation rate r and the decomposition factor α .

Over a long period of time, the leaf mass approaches an equilibrium value M . Find M in terms of the annual accumulation rate r and the decomposition factor α .

The equilibrium solution occurs when $dm/dt = 0$. So: $dm/dt = 0 = r - \alpha M$ and

$$M = \frac{r}{\alpha} \quad (3 \text{ pts})$$

- (c) (10 pts) A flood washes away all the leaves. How long does it take for the leaf mass to recover to 50% of its equilibrium value?

First solve the differential equation. Rewrite as:

$$\frac{dm}{dt} = -\alpha \left(m - \frac{r}{\alpha} \right) = -\alpha (m - M)$$

Separate variables:

$$\frac{dm}{m - M} = -\alpha dt$$

Integrate:

$$\int \frac{dm}{m - M} = -\alpha \int dt$$

$$\ln |m - M| = -\alpha t + C$$

$$m(t) = M + Ae^{-\alpha t}$$

Initially all the leaves are washed away, so $m(0) = 0$:

$$m(t) = \frac{r}{\alpha}(1 - e^{-\alpha t})$$

Suppose at time $t = T$, the mass of leaves is $M/2 = r/(2\alpha)$. Then:

$$m(T) = \frac{1}{2} \frac{r}{\alpha} = \frac{r}{\alpha}(1 - e^{-\alpha T})$$

$$\frac{1}{2} = 1 - e^{-\alpha T}$$

$$e^{-\alpha T} = \frac{1}{2}$$

$$-\alpha T = \ln \frac{1}{2}$$

$$T = \frac{1}{\alpha} \ln 2$$

(10 pts)

8. (10 Points) A paratrooper bails out of an airplane at a height of 1.5 km and opens her parachute at a height of 0.5 km above the ground. It is known that the limiting velocity of a free falling human body is 50 meters per second. Take air resistance to be proportional to the square of the velocity. [In this problem you may assume that gravitational constant is 10 m/s^2 .]
- (a) (2 pts) Write down the differential equation governing the dynamics of the paratrooper.
 - (b) (5 pts) Solve the differential equation.
 - (c) (3pts) How much time did she spend in the air before opening the parachute?

9. (10 Points)

(a) (6 pts) Find the general solution of

$$y^{(4)} + 2y^{(3)} + y'' + 4y' + 4y = 0.$$

In this case characteristic equation is

$$r^4 + 2r^3 + r^2 + 4r + 4 = r^3(r + 2) + (r + 2)^2 = (r + 2)(r^3 + r + 2) = 0.$$

To factor cubic polynomial we find $r = -1$ by inspection, then use long division to get

$$(r + 2)(r + 1)(r^2 - r + 2) = 0.$$

(b) (4 pts) Given that $y(0) = -1$, $y'(0) = y''(0) = y^{(3)}(0) = 1$, solve the initial value problem.

10. (10 Points)

(a) (6 pts) Using Wronskian decide whether the following three functions are linearly independent

$$f(x) = e^x, g(x) = e^{-x}, h(x) = e^{-2x}.$$

(b) (4 pts) Solve the following initial value problem

$$y^3 + 2y'' - y' - 2y = 0, y(0) = 1, y'(0) = 2, y''(0) = 0,$$

given that f , g , and h from part (a) are some solutions to the differential equation.

Table of Integrals

BASIC FORMS

- (1) $\int x^n dx = \frac{1}{n+1} x^{n+1}$
- (2) $\int \frac{1}{x} dx = \ln x$
- (3) $\int u dv = uv - \int v du$
- (4) $\int u(x)v'(x) dx = u(x)v(x) - \int v(x)u'(x) dx$

RATIONAL FUNCTIONS

- (5) $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln(ax+b)$
- (6) $\int \frac{1}{(x+a)^2} dx = \frac{-1}{x+a}$
- (7) $\int (x+a)^n dx = (x+a)^n \left(\frac{a}{1+n} + \frac{x}{1+n} \right), n \neq -1$
- (8) $\int x(x+a)^n dx = \frac{(x+a)^{n+1}(nx+x-a)}{(n+2)(n+1)}$
- (9) $\int \frac{dx}{1+x^2} = \tan^{-1} x$
- (10) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}(x/a)$
- (11) $\int \frac{x dx}{a^2+x^2} = \frac{1}{2} \ln(a^2+x^2)$
- (12) $\int \frac{x^2 dx}{a^2+x^2} = x - a \tan^{-1}(x/a)$
- (13) $\int \frac{x^3 dx}{a^2+x^2} = \frac{1}{2} x^2 - \frac{1}{2} a^2 \ln(a^2+x^2)$
- (14) $\int (ax^2+bx+c)^{-1} dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \left(\frac{2ax+b}{\sqrt{4ac-b^2}} \right)$
- (15) $\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} [\ln(a+x) - \ln(b+x)], a \neq b$
- (16) $\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln(a+x)$
- (17) $\int \frac{x}{ax^2+bx+c} dx = \frac{\ln(ax^2+bx+c)}{2a} - \frac{b}{a\sqrt{4ac-b^2}} \tan^{-1} \left(\frac{2ax+b}{\sqrt{4ac-b^2}} \right)$

INTEGRALS WITH ROOTS

- (18) $\int \sqrt{x-a} dx = \frac{2}{3} (x-a)^{3/2}$
- (19) $\int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a}$
- (20) $\int \frac{1}{\sqrt{a-x}} dx = 2\sqrt{a-x}$
- (21) $\int x\sqrt{x-a} dx = \frac{2}{3} a(x-a)^{3/2} + \frac{2}{5} (x-a)^{5/2}$
- (22) $\int \sqrt{ax+b} dx = \left(\frac{2b}{3a} + \frac{2x}{3} \right) \sqrt{b+ax}$
- (23) $\int (ax+b)^{3/2} dx = \sqrt{b+ax} \left(\frac{2b^2}{5a} + \frac{4bx}{5} + \frac{2ax^2}{5} \right)$
- (24) $\int \frac{x}{\sqrt{x \pm a}} dx = \frac{2}{3} (x \pm 2a) \sqrt{x \pm a}$
- (25) $\int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x} \sqrt{a-x} - a \tan^{-1} \left(\frac{\sqrt{x} \sqrt{a-x}}{x-a} \right)$
- (26) $\int \sqrt{\frac{x}{x+a}} dx = \sqrt{x} \sqrt{x+a} - a \ln \left[\sqrt{x} + \sqrt{x+a} \right]$
- (27) $\int x\sqrt{ax+b} dx = \left(-\frac{4b^2}{15a^2} + \frac{2bx}{15a} + \frac{2x^2}{5} \right) \sqrt{b+ax}$
- (28) $\int \sqrt{x} \sqrt{ax+b} dx = \left(\frac{b\sqrt{x}}{4a} + \frac{x^{3/2}}{2} \right) \sqrt{b+ax} - \frac{b^2 \ln(2\sqrt{a}\sqrt{x} + 2\sqrt{b+ax})}{4a^{3/2}}$
- (29) $\int x^{3/2} \sqrt{ax+b} dx = \left(-\frac{b^2\sqrt{x}}{8a^2} + \frac{bx^{3/2}}{12a} + \frac{x^{5/2}}{3} \right) \sqrt{b+ax} - \frac{b^3 \ln(2\sqrt{a}\sqrt{x} + 2\sqrt{b+ax})}{8a^{3/2}}$
- (30) $\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} x\sqrt{x^2 \pm a^2} \pm \frac{1}{2} a^2 \ln \left(x + \sqrt{x^2 \pm a^2} \right)$
- (31) $\int \sqrt{a^2 - x^2} dx = \frac{1}{2} x\sqrt{a^2 - x^2} - \frac{1}{2} a^2 \tan^{-1} \left(\frac{x\sqrt{a^2 - x^2}}{x^2 - a^2} \right)$
- (32) $\int x\sqrt{x^2 \pm a^2} = \frac{1}{3} (x^2 \pm a^2)^{3/2}$
- (33) $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left(x + \sqrt{x^2 \pm a^2} \right)$