1. Objectives and Instructions

1.1. Model. In this lab we are going to study a system of differential equations which models the excitation of atoms in a lasing medium. Essentially, we will be learning (some of) the mathematics behind how lasers work. Atomic physics is, of course, outside of the scope of this course, but the modeling equations are ones we are able to analyze.

The system we will be analyzing is

\[
\begin{align*}
N' &= \gamma (A - N(1 + P)) \\
P' &= P(N - 1)
\end{align*}
\]

where \( P \) is the laser's intensity and \( N \) is the population inversion function for atoms in the laser, and \( \gamma \) and \( A \) are constants. This is a nonlinear system which we cannot solve analytically, but we will see that for large \( t \), the intensity \( P \) is similar to the solution of a second order equation for a damped vibration, and we will explore resonance in this context.

1.2. Objectives. Our goals for this lab are to extend our use of matrix methods and to further investigate the behavior of solutions to linear and nonlinear equations. In particular, we will:

- see relationships and differences between first-order linear and nonlinear systems; and
- investigate damped harmonic oscillations and resonance.

Note that the first of these looks at systems of equations, while to look at the forcing that gives resonance, as we do in the second, we will look at an equivalent single, higher-order differential equation. We have already seen that a single higher-order differential equation can be written as a system to provide insight on the differential equation; in this lab we also make the transformation of a system into an equivalent second-order differential equation to provide insight on the system.

2. Pre-Lab

In this Pre-Lab, we will see in a general sense where (1) comes from. The three key things we need to know are: 1. that the atoms in the lasing medium

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What “intensity” and “population inversion” mean is explained in §2.2. “Intensity” is essentially what you expect, and “population inversion” is a measure of the energy in the system to produce laser light.
(ruby, for our work here) can be in one of three energy states; 2. that we can add energy to the laser, so that we move atoms to a higher energy state; and 3. that atoms tend to decay to lower energy states, and the energy released by the decay can manifest as a released photon—and when the released photons can trigger the release of more photons, we get “lasing,” which is the release of coherent, laser light.

2.1. Energy levels. In a ruby laser, each atom of the ruby is in one of three energy states, which we will refer to as $E_1$, $E_2$, and $E_3$. $E_1$ is the lowest energy level (the ground state), and $E_3$ the highest energy. Let $n_1(t)$, $n_2(t)$, and $n_3(t)$ be the number of atoms in each energy level at time $t$. Atoms in the $E_2$ and $E_3$ states will naturally decay to a lower level by releasing energy, so that without external stimulation all atoms will eventually fall to the ground state $E_1$. The laser adds an “energy pump” that moves atoms from state $E_1$ to state $E_3$ and increases the tendency of atoms in state $E_3$ to decay to $E_1$. The combination of the decay and energy pump means that $n_1$, $n_2$ and $n_3$ satisfy the system

\[
\begin{align*}
n_1' &= -W_p n_1 + \gamma_{21} n_2 + (\gamma_{31} + W_p) n_3 \\
n_2' &= -\gamma_{21} n_2 + \gamma_{32} n_3 \\
n_3' &= W_p n_1 - (\gamma_{32} + \gamma_{31} + W_p) n_3,
\end{align*}
\]

or, in matrix form with $n = (n_1 \ n_2 \ n_3)^T$,

\[
n' = (G + W)n = \begin{pmatrix} 0 & \gamma_{21} & \gamma_{31} \\ 0 & -\gamma_{21} & \gamma_{32} \\ 0 & 0 & -\gamma_{32} - \gamma_{31} \end{pmatrix} + \begin{pmatrix} -W_p & 0 & W_p \\ 0 & 0 & 0 \\ W_p & 0 & -W_p \end{pmatrix} n
\]

\[
= \begin{pmatrix} -W_p & \gamma_{21} & \gamma_{31} + W_p \\ 0 & -\gamma_{21} & \gamma_{32} \\ W_p & 0 & -\gamma_{32} - \gamma_{31} - W_p \end{pmatrix} n.
\]

Here the matrix $G$ captures the effect of spontaneous decay between energy states: the $\gamma_{ij}$ are the (constant, positive) rates at which atoms spontaneously decay from level $i$ to level $j$. And the matrix $W$ gives the effect of the energy pump.

**Example 1: Why are the entries in the second row and column of $W$ all zero?**

The matrix $W$ models the effect of the energy pump, which moves atoms from state $E_1$ to state $E_3$ and stimulates the release of energy from atoms in state $E_3$ so that the drop to state $E_1$. It doesn’t have any effect on atoms in state $E_2$. The second column gives the effect of atoms in state $n_2$ on those in states $n_1$ and $n_3$, and the second row the change in $n_2$ as a result of the energy pump. Both of these need to be zero for the pump to behave as advertised.
Exercise 1: Why does the first column of $G$ contain only zeros? Why do some $\gamma_{ij}$ have a minus sign and the others a plus sign?

In the case of ruby lasers, the spontaneous emission rates are $\gamma_{21} = \frac{1}{3}$ ms$^{-1}$, and $\gamma_{32} = \gamma_{31} = 10^4$ ms$^{-1}$. Note that not only is $\gamma_{32} > \gamma_{21}$, but that it is much larger. Physically, this means is that atoms in energy level $E_3$ decay almost immediately to $E_2$ or $E_1$, and relative to this, atoms in level $E_2$ decay slowly to $E_1$. This discrepancy in behavior between the energy levels is necessary for lasers to exist at all.

2.2. Stimulated emission and lasers. Finally, we need to introduce the photons that make the laser. (The word LASER is an acronym for “Light Amplification by Stimulated Emission of Radiation.”) When an atom changes from a state $E_2$ to $E_1$ it releases energy, which may take the form of a photon. The “Amplification” in the “LASER” begins when photons that are emitted by this state change circulate through the lasing medium (the ruby), interact with another atom of energy $E_2$ and stimulate it to change to state $E_1$ and emit another photon of the same frequency. This is called “Lasing.” Note that lasing has the effect of increasing the number of photons in the system, and decreasing the number of atoms in energy level $E_2$.

To model this, we rewrite the system (3) to account for this decrease, and add an equation for the number of lasing photons in the system. Letting $p(t)$ be this number of photons, the equation for $p$ turns out to be

$$p' = p(-\gamma_c + K(n_2 - n_1)),$$

where $K$ is a positive constant called the gain rate, which represents the increase in lasing photons because of the photons’ interaction with atoms in state $E_2$ to stimulate release of more photons, and $\gamma_c$ is the rate at which lasing photons leave the system entirely (as laser light!). Note that this equation is in terms of $n_2 - n_1$. We will call this the “population inversion function” in a moment.

To rewrite (3), note that because the atoms at energy level $E_3$ decay much faster than those at $E_2$ we might get away with the assumption that $n_3(t) = 0$. If we take $n_3(t) = 0$ and rewrite (3) in terms of the “population inversion function” $n(t) = n_2(t) - n_1(t)$, we can rewrite the system as a single equation (you do this in Exercise 2).

Example 2: Let $n_T$ be the total number of atoms in the laser, so that $n_T = n_1 + n_2 + n_3$. Derive expressions for $n_1$ and $n_2$ in terms of $n$ and $n_T$, assuming that $n_3(t) = 0$.

Because $n_3 = 0$, we have $n_T = n_1 + n_2$. By definition, $n = n_2 - n_1$. If we add these two expressions, we get $2n_2 = n_T + n$, so that $n_2 = \frac{1}{2}(n_T + n)$. If we subtract them, we get $2n_1 = n_T - n$, so that $n_1 = \frac{1}{2}(n_T - n)$.

Exercise 2: Rewrite system (3) as a single equation in $n$, by assuming that $n_3 = 0$ and subtracting the remaining equations to get an equation for $n' = (n_2 - n_1)'$. (You will need the results derived in Example 2.)
Finally, the lasing photons reduce the number of atoms in state \( E_2 \), so we have to add a term to the equation that you found in exercise 2, which gives the nonlinear system

\[
\begin{align*}
n' &= -\left(\frac{1}{2} W_p + \gamma_{21}\right)n + \left(\frac{1}{2} W_p - \gamma_{21}\right)n_T - 2Kn_p \\
p' &= p(-\gamma_c + Kn)
\end{align*}
\]

It is convenient to rewrite (4) so that time is measured as a number of photon decay periods, and so that the population inversion function and number of lasing photons are measured as fractions of various equilibrium values. We omit the details of how that is done; the resulting, simplified, system is that which we introduced as our model at the beginning of the lab,

\[
\begin{align*}
N' &= \gamma(A - N(1 + P)) \\
P' &= P(N - 1).
\end{align*}
\]

We refer to the scaled variable \( N \) as the population inversion, and call \( P \) the intensity function. The constants \( \gamma \) and \( A \) are combinations of the other constants in the problem (it happens that \( \gamma = \frac{\frac{1}{2} W_p + \gamma_{21}}{\gamma_c} \) and \( A = \frac{\left(\frac{1}{2} W_p - \gamma_{21}\right)KNT}{\left(\frac{1}{2} W_p + \gamma_{21}\right)\gamma_c} \)).

We can think of \( A \) as a measure of how efficient the laser is (that is, how much it intensity increases as the population inversion increases) and \( \gamma \) as a measure of how lasing photons build up in the laser (that is, a ratio of photon creation to photon release as laser light). We will assume that these are positive in all lab exercises.

**Exercise 3:** Find the critical points of (5).

**Exercise 4:** The linearization of (5) at \((1, A - 1)\) is

\[
\begin{align*}
u' &= -\gamma(Au + v) \\
v' &= (A - 1)u
\end{align*}
\]

In this exercise we see how we can obtain this.

(a) Let \( N = 1 + u \) and \( P = (A - 1) + v \). Plug these into the system (5) and expand. If \( u \) and \( v \) are both very small in magnitude, so that we can discard nonlinear terms, show that you obtain the linearized system above.

(b) Plot \((1, A - 1)\) in the phase plane. What do \( u \) and \( v \) measure in the phase plane? The function \( P_{RO}(t) = v(t) + A - 1 \) is called the relaxation oscillation (RO) of the laser. How is it related to \( P \)?

(c) Rewrite the linear system as a single second order linear equation for \( v \).

**References**
