LAB 6: NONLINEAR SYSTEMS AND NEURON MODELING
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1. INTRODUCTION

This lab is different from the other five labs, being more similar to Lab 0 than Labs 1–5. You will complete all of the work for this lab in pairs, with a partner, and should be able to complete it in the course of the lab period.

Our goals in this lab are to work with a new nonlinear system to:

• See how our linear analysis (still!) works to give some sense of the behavior of the system,
• Explore the use of nullclines to get a qualitative, but still analytical, understanding of the nonlinear dynamics, and
• Explore the nonlinear behavior numerically to confirm the insights that we get from the work above.

For this lab, you will be doing an informal reflection. The goals of this are for you to better understand the analysis of nonlinear systems such as (1), below, and to see connections between this analysis and other work that we have done in the course. The details of this reflection are at the end of this lab document.

2. MATLAB

MATLAB commands we use in this lab include the following.

2.1. Neuron Model. This isn’t a native MATLAB command; download it from the labs page. See the Reflection section, below, for information about this.

2.2. ode45. Finds a numerical approximation to a differential equation or system of equations:

\[
[tsol, xsol] = \text{ode45}(f\_\text{handle}, [tmin tmax], \text{init}\_\text{cond});
\]

2.3. plot. Plot one vector against another; e.g.,

\[
\text{plot}(\ tsol, \ xsol(:,1)) ;
\]

3. BACKGROUND

In this lab we consider a very simple model for the behavior of a neuron, called the Fitzhugh-Nagumo model. The model is

\[
\begin{align*}
\tau v' &= v - \frac{1}{3} v^3 - w + I_{\text{ext}} \\
\tau w' &= v + a - bw,
\end{align*}
\]

in which \(v\) may be identified with the voltage across the membrane of an axon (the long, slender projection of a nerve cell, or neuron, that conducts electrical impulses away from the nerve cell body), and \(w\) a measure of the forces in the neuron.
cell that return it to rest. The parameter $I_{\text{ext}}$ is an external stimulus (voltage) applied to the neuron—which could arise from an electrode, or could be the result of another neuron firing. For our analysis we will consider $I_{\text{ext}}$ to be constant. The other parameters in the system are $a$, $b$, and $\tau$, which model other characteristics of the physical system being modeled (but which we do not explore here).

4. Exercises

Exercise 1. The first step in the analysis of a nonlinear system such as (1) is to find the critical points. Note that in this case doing so will involve solving a cubic equation. Yuck. Instead of trying to do that by hand, let’s use MATLAB to plot the nullclines—where the derivatives $v'$ and $w'$ are zero. Suppose that $I_{\text{ext}}$ is zero—what equation (in $w$ and $v$) must be satisfied if $v' = 0$? In the phase plane we’ll think of $v$ as our $x$ variable and $w$ as our $y$ variable. Show that $v' = 0$ wherever $w = v - v^3/3$. This is the v-nullcline.

Similarly find where $w' = 0$ (the w-nullcline). Take $a = 0.7$ and $b = 0.8$ and plot the two nullclines using MATLAB, with commands something like the following:

```matlab
>> vvals = -2:.05:2;
>> vnull = vvals - vvals.^3/3;
>> wnull = ...
>> plot(vvals,vnull,'-b', vvals,wnull,'-m', 'LineWidth',2);
```

Given your graph of the two nullclines, where is the critical point for the system? *(Hint: what are $v'$ and $w'$ at the critical point?)* We will call the critical point $(v_0, w_0)$ in the following. What happens to the $v$ coordinate of the critical point, $v_0$, as the external stimulus $I_{\text{ext}}$ increases from zero? *(Hint: if this isn’t obvious, you can try graphing the nullclines with several values of $I_{\text{ext}}$ to see.)*

Exercise 2. Find the Jacobian for the system (1). Note that if we use the values of $a$ and $b$ given above, and specify a value for $\tau$—which we’ll take as $\tau = 12.5$—then the Jacobian, and hence its eigenvalues, and hence the behavior of the critical point, depend on the value of $v_0$. And $v_0$ depends on $I_{\text{ext}}$, as we found in Exercise 1. To see what the predicted linear behavior is, use MATLAB’s eig command to find the eigenvalues of the Jacobian when $v_0 = -1.25$, $v_0 = -1$, $v_0 = -0.95$, $v_0 = -0.75$, and $v_0 = -0.5$. What will the phase portrait look like at the critical point in each of these cases? Sketch a rough phase portrait near the critical point for each of these values of $v_0$—for the last, work out what you think will happen based on the eigenvalues and the behavior of the first ones. *(Hint: you may want to find the direction of motion by using the Jacobian.)*

Exercise 3. What does your linear analysis tell you about the behavior of the nonlinear system? In Exercise 2 we looked at the behavior as a function of
v₀, which is determined by the parameter we would actually change in the real-world, I_ext. We can't get a useful analytical expression that gives v₀ as a function of I_ext, but solving numerically we can determine the values shown below:

<table>
<thead>
<tr>
<th>I_ext</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>v₀</td>
<td>−1.199</td>
<td>−1.138</td>
<td>−1.069</td>
<td>−0.993</td>
<td>−0.907</td>
<td>−0.805</td>
</tr>
<tr>
<td>w₀</td>
<td>−0.624</td>
<td>−0.547</td>
<td>−0.462</td>
<td>−0.367</td>
<td>−0.258</td>
<td>−0.131</td>
</tr>
</tbody>
</table>

As we increase I_ext, at which value(s) do we expect to see the behavior of the nonlinear system change? Suppose I_ext = 0.3; sketch by hand what you think the nonlinear phase portrait will look like. Then suppose that I_ext = 0.5 and generate a similar sketch. For each, also sketch what v(t) will look like.

**Exercise 4.** Finally, let's use MATLAB to see how the nonlinear system actually behaves. For which values of I_ext is this most necessary?

Consider I_ext = 0.3, and then I_ext = 0.5. In each case, generate solutions using ode45 for several different initial conditions, and then plot the phase portrait and a component plot of v(t) vs. t.

Note that the periodic behavior you see indicates that with a sufficiently large external stimulus, we expect the neuron to “fire” repeatedly, sending a signal through the axon to other neurons. This is the behavior that we associate with brain activity: neurons are excited and generate an electrical signal that may elicit activity in other neurons.

**5. Reflection**

This is more informal than the reflections that you completed for labs 2 and 4, in that it serves to help you think through what you’ve just investigated and will hand it in as noted below. To help think about what you’ve just done, download the MATLAB file Neuron_Model.m from the lab page. Notice that at the top of the file are a number of configuration variables; you should be able to leave those largely unchanged, though you may want to play with the speed or tail parameters if the animation is taking a long time to run. Then run the script; it should show you much of what you found in Exercise 4... though likely with a little more pizzaz. After you’ve done that, consider the following. Take two minutes to think about the answers to these questions yourself, and then talk with the others at your table to be sure that you all have a clear and consistent understanding of the answers.

(1) When we are analyzing a nonlinear system, we look for critical points and then analyze their stability. *When* does this analysis tell us something about what we expect to see in the nonlinear system?

(2) Conversely, if we are not in the situation you note in (1), what can we say about the nonlinear system? (*Your answer to this will be very open: you can say something about what you don’t expect to see...*)
(3) The graphs that you obtained in this lab should remind you of your work in another lab. Where have you seen similar behavior? In fact, if you set $a = b = 0$ in the Fitzhugh-Nagumo equations (1), you will get a slightly rescaled van der Pol oscillator.

(a) What effect did we say in lab 2 resulted in the limit cycle?

(b) Recall that in lab 5 we also saw that the nonlinear trajectories were constrained to an attracting trajectory, albeit a more complicated one. Explain how your work in labs 2, 5, and this lab are related, and how the linearization and solution of the linear problem is helpful in determining the expected behavior of the system.

Write up a brief summary of your responses to these questions and submit it in Canvas as your report for this lab.

References