Text: Section 3.5, Practice problems # 1, 3, 5, 7, 9
We consider a system of equations

\[ \begin{align*}
    x'(t) &= ax(t) + by(t) \\
    y'(t) &= cx(t) + dy(t),
\end{align*} \]  \hspace{1cm} (1)

It may happen that the matrix

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

has just one eigenvalue, that is the equation

\[ \lambda^2 - \lambda(a + d) + (ad - bc) = 0 \]

has a single root \( \lambda \).
The case of repeated eigenvalues

Example

Suppose that

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]

so the equation is

\[ \lambda^2 - 2\lambda + 1 = 0 \]

and \( \lambda = 1 \). The system reads

\[
\begin{align*}
    x'(t) &= x(t) \\
y'(t) &= y(t),
\end{align*}
\]

And the general solution is

\[ x(t) = C_1 e^t \quad \text{and} \quad y(t) = C_2 e^t \]

for some constants \( C_1 \) and \( C_2 \).
The easy case

We get a simple answer in the above example because although we had a single eigenvalue, we had plenty of essentially different eigenvectors. In the exercises below, we assume that $A$ is a $2 \times 2$ matrix with eigenvalue $\lambda$ and eigenvectors $\vec{v}_1$ and $\vec{v}_2$, none of which is a scalar multiple of the other. Let

$$\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

**Exercise:** Show that

$$\det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \neq 0.$$

**Exercise:** Show that for any 2-vector $\vec{v}$, one can find constants $c_1$ and $c_2$, such that

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2.$$
The easy case

Exercise: Show that for any 2-vector \( \vec{v} \), we have

\[ A\vec{v} = \lambda \vec{v}. \]

Exercise: Show that

\[ A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}. \]

Exercise: Sketch typical trajectories for the system

\[ \begin{align*}
    x'(t) &= \lambda x(t) \\
y'(t) &= \lambda y(t)
\end{align*} \]

when a) \( \lambda > 0 \), b) \( \lambda < 0 \) and c) \( \lambda = 0 \).
The hard case

Sometimes, however, we don’t have enough eigenvectors.

Example

Let

\[ A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}. \]

We to find the eigenvalues, we solve the equation

\[ 0 = \det \begin{bmatrix} 2 - \lambda & 1 \\ -1 & 4 - \lambda \end{bmatrix} = (2-\lambda)(4-\lambda) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2, \]

so the only eigenvalue is \( \lambda = 3 \).

We find the eigenvectors

\[ \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \]
The hard case

Example (Finished)

by solving the equation

$$(A - \lambda I)\vec{v} = \vec{0} \implies \begin{bmatrix} 2 - 3 & 1 \\ -1 & 4 - 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

that is,

$$-x + y = 0$$
$$-x + y = 0,$$

from which we can pick an eigenvector

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

but any other eigenvector is a scalar multiple of this one (Exercise: why?).
For the system of equations

\[
\begin{align*}
x'(t) &= 2x(t) + y(t) \\
y'(t) &= -x(t) + 4y(t),
\end{align*}
\]

we get solutions

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = Ce^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{so that} \quad \begin{align*}
x(t) &= Ce^{3t} \\
y(t) &= Ce^{3t},
\end{align*}
\]

where $C$ is an arbitrary constant.

These, however, are not all solutions.

**Exercise:** Find some initial conditions for which the system (2) does not have solutions of the type (3).

We first describe a procedure to find all solutions, and then we explain the intuition behind the procedure.
The Procedure: Consider a system of linear equations (1) and suppose that the matrix

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

has a unique eigenvalue \( \lambda \). Suppose further, that up to scaling \( \vec{v} \mapsto \alpha \vec{v} \), there is a unique eigenvector \( \vec{v} \) with eigenvalue \( \lambda \). One basic solution is found as

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \vec{v}.
\]

We look for another solution of the system (1) in the form

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = te^{\lambda t} \vec{v} + e^{\lambda t} \vec{w}, \tag{4}
\]

where \( \vec{w} \) is a certain vector, yet to be found.
Substituting (4) into (2), we compute

\[
\begin{bmatrix}
x'(t) \\
y'(t)
\end{bmatrix} = e^{\lambda t} \vec{v} + \lambda te^{\lambda t} \vec{v} + \lambda e^{\lambda t} \vec{w}
\]

and

\[
A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = te^{\lambda t} A\vec{v} + e^{\lambda t} A\vec{w} = \lambda te^{\lambda t} \vec{v} + e^{\lambda t} A\vec{w},
\]

so we must have

\[
A\vec{w} = \vec{v} + \lambda \vec{w}.
\]

It turns out that we can always find \( \vec{w} \) satisfying (5). Then the general solution to (2) is

\[
C_1 e^{\lambda t} \vec{v} + C_2 \left( te^{\lambda t} \vec{v} + e^{\lambda t} \vec{w} \right)
\]

for arbitrary constants \( C_1 \) and \( C_2 \).
The procedure

Example

We consider the system of equations (2). For the matrix

\[ A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}, \]

there is a unique eigenvalue \( \lambda = 3 \) and a unique, up to scaling, eigenvector

\[ \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

We look for a vector

\[ \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \]

such that

\[ \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \]
The procedure

Example (Continued)

that is,

\[
\begin{align*}
2w_1 + w_2 &= 1 + 3w_1 \\
-w_1 + 4w_2 &= 1 + 3w_2 \\
\end{align*}
\Rightarrow -w_1 + w_2 &= 1
\]

so we can choose

\[
\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Hence the general solution to the system is

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \left( te^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),
\]

that is,

\[
x(t) = C_1 e^{3t} + C_2 te^{3t} \\
y(t) = C_1 e^{3t} + C_2 (te^{3t} + e^{3t})
\]
The procedure

Example (Finished)

for an arbitrary constants \( C_1 \) and \( C_2 \).

For example, if we want to satisfy the initial conditions \( x(0) = 1 \) and \( y(0) = 2 \), we get \( C_1 = 1 \) and \( C_2 = 1 \), so

\[
\begin{align*}
  x(t) &= e^{3t}(t + 1) \\
  y(t) &= e^{3t}(t + 2).
\end{align*}
\]

Exercise: Find the solution to the system

\[
\begin{align*}
  x'(t) &= 3x(t) - y(t) \\
  y'(t) &= x(t) + y(t)
\end{align*}
\]

satisfying the initial conditions \( x(0) = 1 \) and \( y(0) = 0 \).
The intuition

Next, we give an intuitive explanation of what is going on here. It is important to note that the condition that the matrix has a unique eigenvalue is “fragile”.

**Exercise:** Show that for all sufficiently small $r > 0$, the matrix

$$A(r) = \begin{bmatrix} 2 & 1 \\ -1 + r & 4 \end{bmatrix}$$

has two distinct real eigenvalues.

**Exercise:** Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a matrix with a unique eigenvalue $\lambda$. Show that one can find a matrix

$$\tilde{A} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix}$$

with two distinct real eigenvalues $\lambda_1 > \lambda_2$, where the entries $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are arbitrarily close to the corresponding entries $a, b, c, d$. 
The intuition

Now, suppose that $A$ has a unique eigenvalue $\lambda$ and unique, up to scaling, eigenvector $\vec{v}$. For a small $r > 0$ we consider a perturbation $A(r)$ of $A$ such that $A(r)$ has two distinct real eigenvalues $\lambda_1(r) > \lambda_2(r)$ with eigenvectors $\vec{v}_1(r)$ and $\vec{v}_2(r)$ and

$$A(r) \longrightarrow A \quad \text{as} \quad r \longrightarrow 0^+.$$ 

We of course expect that

$$\lambda_1(r), \lambda_2(r) \longrightarrow \lambda \quad \text{and} \quad \vec{v}_1(r), \vec{v}_2(r) \longrightarrow \vec{v}$$

as $r \longrightarrow 0^+$. Since

$$e^{\lambda_1(r)t} \vec{v}_1(r) \quad \text{and} \quad e^{\lambda_2(r)t} \vec{v}_2(r)$$

are solution of the perturbed system with matrix $A(r)$, by taking the limit as $r \longrightarrow 0^+$, we conclude that

$$e^{\lambda t} \vec{v}$$

is a solution to the system with matrix $A$. 
The intuition

However, the functions

\[ \frac{1}{r} \left( e^{\lambda_1(r) t} \vec{v}_1(r) - e^{\lambda_2(r) t} \vec{v}_2(r) \right) \]

will also be a solution to the perturbed system with matrix \( A(r) \) (Exercise: Why?)

Furthermore, we can write

\[ \frac{1}{r} \left( e^{\lambda_1(r) t} \vec{v}_1(r) - e^{\lambda_2(r) t} \vec{v}_2(r) \right) = \frac{e^{\lambda_1(r) t} \vec{v}_1(r) - e^{\lambda_2(r) t} \vec{v}_2(r)}{r} - \frac{e^{\lambda_1(r) t} \vec{v}_1(r) - e^{\lambda_2(r) t} \vec{v}_2(r)}{r} \]

Taking the limit as \( r \to 0^+ \), we conclude that

\[ \frac{d}{dr} \left( e^{\lambda_1(r) t} \vec{v}_1(r) \right) - \frac{d}{dr} \left( e^{\lambda_2(r) t} \vec{v}_2(r) \right) \] at \( r = 0 \)

(we take the right derivative here) is a solution to the original system with matrix \( A \).
Differentiating, we get that

$$\lambda_1'(0)t e^{\lambda t} \vec{v} + e^{\lambda t} \vec{v}_1'(0) - \lambda_2'(0)t e^{\lambda t} \vec{v} - e^{\lambda t} \vec{v}_2'(0)$$

is a solution to the original system of linear equations with matrix $A$. Hence for the mysterious vector $\vec{w}$, we have

$$\vec{w} = \frac{1}{\lambda_1'(0) - \lambda_2'(0)} (\vec{v}_1'(0) - \vec{v}_2'(0)).$$
Pictures of trajectories

**Exercise:** What goes wrong with our explanation if

\[ A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \]?

Next, we draw some pictures of trajectories.
Suppose that \( \lambda > 0 \).
After scaling, the general solution can be written as

\[
\begin{bmatrix}
  x(t) \\
  y(t)
\end{bmatrix} = te^{\lambda t} \vec{v} + e^{\lambda t} \vec{w} + Ce^{\lambda t} \vec{v}.
\]

For \( t \gg 0 \), we have \( e^{\lambda t} \gg 0 \) and

\[
\begin{bmatrix}
  x(t) \\
  y(t)
\end{bmatrix} \approx te^{\lambda t} \vec{v}.
\]
For $t \ll 0$, we have $te^{\lambda t}, e^{\lambda t} \approx 0$ and

\[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} \approx te^{\lambda t} \vec{v}.
\]

Hence typical trajectories look like this:

It is called an \textit{unstable improper node}. 
Exercise: Draw pictures of trajectories for $\lambda < 0$ and $\lambda = 0$. 