Text: Section 3.4, Practice problems # 1, 3, 5, 7, 15
The case of two complex eigenvalues

We consider the system

\begin{align*}
x'(t) &= ax(t) + by(t) \\
y'(t) &= cx(t) + dy(t)
\end{align*}

and assume that the equation

\[ \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \]

has two distinct complex solutions

\[ \lambda = \mu \pm \sigma i \quad \text{where} \quad \sigma \neq 0. \]

The idea is to consider \textit{complex} eigenvectors

\[ \vec{v} = \vec{u} \pm i\vec{w}, \quad \vec{v} \neq \vec{0}, \]

where \( \vec{u} \) and \( \vec{w} \) are real vectors, satisfying the equation

\[ A\vec{v} = \lambda\vec{v}. \]

Exercise: Why can the eigenvectors be written in the form \( \vec{u} \pm i\vec{w} \)?
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We notice that

\[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} = e^{\lambda t} \vec{v}
\] (2)

is still a solution to the system. This, however, is a complex solution and we want real. Next, we observe that both the real and imaginary parts of the complex solution are real solutions of the system (Exercise: why?). The complex solution (2) is written as

\[
e^{(\mu \pm \sigma i)t} (\vec{u} \pm i\vec{w}) = e^{\mu t} (\cos(\sigma t) \pm i \sin(\sigma t)) (\vec{u} \pm i\vec{w})
\]

\[
= e^{\mu t} (\cos(\sigma t) \vec{u} - \sin(\sigma t) \vec{w}) \pm ie^{\mu t} (\cos(\sigma t) \vec{w} + \sin(\sigma t) \vec{u}).
\]
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Hence both

\[
\begin{bmatrix}
    x(t) \\
    y(t)
\end{bmatrix} = e^{\mu t} (\cos(\sigma t) \vec{u} - \sin(\sigma t) \vec{w}) \quad \text{and}
\]

\[
\begin{bmatrix}
    x(t) \\
    y(t)
\end{bmatrix} = \pm e^{\mu t} (\cos(\sigma t) \vec{w} + \sin(\sigma t) \vec{u})
\]

are solutions to (1).

Therefore,

\[
\begin{bmatrix}
    x(t) \\
    y(t)
\end{bmatrix} = e^{\mu t} \left( C_1 (\cos(\sigma t) \vec{u} - \sin(\sigma t) \vec{w}) + C_2 (\cos(\sigma t) \vec{w} + \sin(\sigma t) \vec{u}) \right)
\]

\[
= e^{\mu t} \left( (C_1 \cos \sigma t + C_2 \sin \sigma t) \vec{u} + (-C_1 \sin \sigma t + C_2 \cos \sigma t) \vec{w} \right)
\]

(3)

is also a solution for any constants $C_1$ and $C_2$. 
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In fact, (3) are all solutions of the system (1). We discuss why in the exercises below. There

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

is a real matrix with complex eigenvalues \( \lambda = \mu \pm i\sigma \), where \( \sigma \neq 0 \).

**Exercise:** Suppose \( \vec{v} = \vec{u} \pm i\vec{w} \) are eigenvectors of \( A \), so that \( A\vec{v} = \lambda \vec{v} \). Show that

\[ A\vec{u} = \mu \vec{u} - \sigma \vec{w} \quad \text{and} \quad A\vec{w} = \sigma \vec{u} + \mu \vec{w}. \]

**Exercise:** Use the previous exercise to show that \( \vec{u} \) and \( \vec{w} \) are not scalar multiples of each other.

**Exercise:** Using the previous exercise, show that for any initial conditions \( x(0) = x_0, y(0) = y_0 \), we can find a solution in the form (3), satisfying the system of equations.
Hence we arrive to the following **Procedure**: Suppose that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has complex eigenvalues $\mu \pm \sigma i$ for $\sigma \neq 0$. We compute an eigenvector $\vec{v} = \vec{u} + i\vec{w}$, $\vec{v} \neq \vec{0}$, satisfying the equation

$$A\vec{v} = (\mu + \sigma i)(\vec{u} + i\vec{w}).$$

Then we compute the real and imaginary parts of the expression $e^{(\mu+\sigma i)t}\vec{v}$. The real part multiplied by an arbitrary real constant $C_1$ plus the imaginary part multiplied by an arbitrary real constant $C_2$ is the general solution to the system.
Example

Consider the system

\[
x'(t) = x(t) - y(t) \\
y'(t) = 2x(t) + 3y(t)
\]

The matrix of the system is

\[
A = \begin{bmatrix}
1 & -1 \\
2 & 3
\end{bmatrix}.
\]

We find the eigenvalues by solving the equation

\[
0 = \det \begin{bmatrix}
1 - \lambda & -1 \\
2 & 3 - \lambda
\end{bmatrix} = (1 - \lambda)(3 - \lambda) - (-1) \cdot 2
\]

\[
= \lambda^2 - 4\lambda + 5,
\]
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Example (Continued)

from which the eigenvalues are

\[ \lambda = \frac{4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = 2 \pm i. \]

Alternatively, we could have used the shortcut: the sum of the eigenvalues is the trace \(1 + 3 = 4\), while the product is the determinant \(1 \cdot 3 - (-1) \cdot 2 = 5\). As we suspect that the eigenvalues are complex, \(\mu \pm \sigma i\), we get that \(\mu = 2\) and \(\sigma = 1\). Next, we find an eigenvector with eigenvalue \(2 + i\), say. We look for solutions \[ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \] of the system of equations

\[ \begin{bmatrix} 1 - \lambda & -1 \\ 2 & 3 - \lambda \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]
The case of complex eigenvalues

Example (Continued)

that is,

\[-1 \,-\, i\, \, z_1 \, - \, z_2 \, = \, 0\]
\[2z_1 \, + \, (1 \,-\, i)\, z_2 \, = \, 0.\]

It helps to realize that we have a certain freedom in choosing an eigenvector. For example, if \(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\) is an eigenvector, then for any complex \(\alpha \neq 0\), the vector \(\begin{bmatrix} \alpha z_1 \\ \alpha z_2 \end{bmatrix}\) is also in eigenvector. Therefore, if we can choose \(z_1 \neq 0\), we can choose \(z_1 = 1\) and if we can choose \(z_2 \neq 0\), we can choose \(z_2 = 1\). Let us choose \(z_1 = 1\) and \(z_2 = -1 - i\). It satisfies both equations (Exercise: check).
The case of complex eigenvalues

Example (Continued)

Hence

\[ e^{(2+i)t} \begin{bmatrix} 1 \\ -1 - i \end{bmatrix} = e^{2t} (\cos t + i \sin t) \begin{bmatrix} 1 \\ -1 - i \end{bmatrix} \]

is a complex solution of the system. We take the real part

\[ \begin{bmatrix} e^{2t} \cos t \\ -e^{2t} \cos t + e^{2t} \sin t \end{bmatrix} \]

and the imaginary part

\[ \begin{bmatrix} e^{2t} \sin t \\ -e^{2t} \cos t - e^{2t} \sin t \end{bmatrix} \]
The case of complex eigenvalues

Example (Continued)

and conclude that the general solution is

\[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} = C_1 \begin{bmatrix}
e^{2t} \cos t \\
-e^{2t} \cos t + e^{2t} \sin t
\end{bmatrix} + C_2 \begin{bmatrix}
e^{2t} \sin t \\
-e^{2t} \cos t - e^{2t} \sin t
\end{bmatrix},
\]

that is

\[
x(t) = e^{2t} (C_1 \cos t + C_2 \sin t) \\
y(t) = e^{2t} ((-C_1 - C_2) \cos t + (C_1 - C_2) \sin t)
\]

for arbitrary constants \(C_1\) and \(C_2\).

Suppose we want to satisfy the initial conditions \(x(0) = 1\) and \(y(0) = 1\). We get the equations

\[
\begin{align*}
C_1 & = 1 \\
-C_1 - C_2 & = 1,
\end{align*}
\]
The case of complex eigenvalues

Example (Finished)

from which $C_1 = 1$ and $C_2 = -2$, so the particular solution is

$$x(t) = e^{2t} (\cos t - 2 \sin t)$$
$$y(t) = e^{2t} (\cos t + 3 \sin t)$$

Exercise: Find the solution of the system

$$x'(t) = 3x(t) - 2y(t)$$
$$y'(t) = 4x(t) - y(t)$$

satisfying the initial conditions $x(0) = 1$, $y(0) = 1$.

Next, we draw some pictures. We start with exercises.
Exercise: Sketch the trajectory
\[ x(t) = \cos t, \quad y(t) = \sin t \]
as \( t \) increases from \(-\infty\) to \(+\infty\).

Exercise: Sketch the trajectory
\[ x(t) = e^t \cos t, \quad y(t) = e^t \sin t \]
and the trajectory
\[ x(t) = e^{-t} \cos t, \quad y(t) = e^{-t} \sin t \]
as \( t \) increases from \(-\infty\) to \(+\infty\).
Some observations: If for the eigenvalues $\mu \pm \sigma i$ of the matrix

$$\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}$$

we have $\mu = 0$, then there is no “$e$” part, the equations for $x(t)$ and $y(t)$ involve only $\sin \sigma t$ and $\cos \sigma t$, so we get a periodic trajectory with period $2\pi/|\sigma|$ (Exercise: why?). A typical trajectory looks like an ellipse.

This picture is called a \textit{center}.
The trajectory can move clockwise or counterclockwise, which can be determined by looking at the direction at a couple (often, one) point.

Example

Consider the system

\[
x'(t) = x(t) - 2y(t) \\
y'(t) = x(t) - y(t)
\]

The matrix is

\[
\begin{bmatrix}
1 & -2 \\
1 & -1
\end{bmatrix},
\]

so the eigenvalues are ±i (Exercise: check). We have a center.

If, say, \(x = 1\) and \(y = 0\) then

\[
x' = 1 \quad \text{and} \quad y' = 1
\]
so it looks like the ellipses are traversed counterclockwise.
Example (Finished)

Here is a numerically accurate picture
If for the eigenvalues $\mu \pm i\sigma$, we have $\mu > 0$, we got a *spiral source* (also known as an *unstable focus*).

Again, the spiral may be unwinding clockwise or counterclockwise, which can be determined by looking at the direction at a couple (often, one) point.
Consider the system
\[
x'(t) = x(t) - y(t),
\]
\[
y'(t) = 9x(t) + y(t),
\]
with eigenvalues $1 \pm 3i$ (Exercise: check). We get a spiral source. If, say, $x = 1$ and $y = 0$, then
\[
x' = 1 \quad \text{and} \quad y' = 9,
\]
so the spiral must be unwinding counterclockwise.
Example (Continued)
Example (Finished)

Here is a numerically accurate picture.
If for the eigenvalues $\mu \pm i\sigma$, we have $\mu < 0$, we get a spiral sink (also known as a stable focus).

To figure out which way the spiral is winding down, clockwise or counterclockwise, it suffices to check the direction field in a couple (often, one) point.

Exercise: Sketch typical trajectories and classify them as centers, spiral sources or sinks, for the following systems:
\[ x'(t) = 2x(t) + y(t) \quad x'(t) = x(t) - y(t) \quad \text{and} \]
\[ y'(t) = -5x(t) - 2y(t), \quad y'(t) = 5x(t) - 2y(t), \]
\[ x'(t) = 3x(t) - 2y(t) \]
\[ y'(t) = 5x(t) - 2y(t), \]

Make sure you indicate the direction of the trajectories (clockwise or counterclockwise).