Text: Section 3.2, Practice problems # 1, 9, 23, 25; Section 8.4, Practice problems # 1 a), 3 a)
Euler’s method for higher-order equations

We discuss how to solve numerically a higher-order equation. As usual, we start with an example.

Example

Consider the second-order equation

\[ x''(t) = -x(t) \]

with initial conditions \( x(0) = 0 \) and \( x'(0) = 1 \).

We want to estimate \( x(0.5) \). The trick is to introduce a new dependent variable \( y(t) = x'(t) \) and write the second-order equation as a system of first-order equations

\[ x'(t) = y(t) \]
\[ y'(t) = -x(t). \]

with initial conditions \( x(0) = 0 \) and \( y(0) = 1 \).
Euler’s method for higher-order equations

Example (Continued)

After that, we can proceed in the usual way, although each step gets computationally heavier. Let us apply Euler’s method with 5 steps.

On the first step, we compute:

\[ x'(0) = y(0) = 1, \quad y'(0) = -x(0) = 0, \]
\[ x(0.1) \approx x(0) + x'(0) \cdot 0.1 = 0 + 1 \cdot 0.1 = 0.1 \]
\[ y(0.1) \approx y(0) + y'(0) \cdot 0.1 = 1 + 0 \cdot 0.1 = 1. \]

On the second step, we compute:

\[ x'(0.1) = y(0.1) \approx 1, \quad y'(0.1) = -x(0.1) \approx -0.1, \]
\[ x(0.2) \approx x(0.1) + x'(0.1) \cdot 0.1 \approx 0.1 + 1 \cdot 0.1 = 0.2, \]
\[ y(0.2) \approx y(0.1) + y'(0.1) \cdot 0.1 \approx 1 + (-0.1) \cdot 0.1 = 0.99. \]
Euler’s method for higher-order equations

Example (Continued)

On the third step, we compute:

\begin{align*}
x'(0.2) &= y(0.2) \approx 0.99, \
y'(0.2) &= -x(0.2) \approx -0.2, \\
x(0.3) &\approx x(0.2) + x'(0.2) \cdot 0.1 \approx 0.2 + 0.99 \cdot 0.1 = 0.299, \\
y(0.3) &\approx y(0.2) + y'(0.2) \cdot 0.1 \approx 0.99 + (-0.2) \cdot 0.1 = 0.97.
\end{align*}

On the fourth step, we compute:

\begin{align*}
x'(0.3) &= y(0.3) \approx 0.97, \
y'(0.3) &= -x(0.3) \approx -0.299, \\
x(0.4) &\approx x(0.3) + x'(0.3) \cdot 0.1 \approx 0.299 + 0.97 \cdot 0.1 = 0.396, \\
y(0.4) &\approx y(0.3) + y'(0.3) \cdot 0.1 \approx 0.97 + (-0.299) \cdot 0.1 = 0.9401.
\end{align*}
Example (Finished)

Finally, we compute:

\[ x'(0.4) = y(0.4) \approx 0.9401 \quad \text{and} \quad x(0.5) \approx x(0.4) + x'(0.4) \cdot 0.1 \approx 0.396 + 0.9401 \cdot 0.1 = 0.49001. \]

Hence Euler’s method gives:

\[ x(0.5) \approx 0.49001. \]

In fact, the solution is \( x(t) = \sin t \) and

\[ x(0.5) = \sin 0.5 \approx 0.4794255386. \]
Euler’s method for higher-order equations

It is pretty clear now that the method can be applied to any equation of the type

\[ x''(t) = f(t, x(t), x'(t)) \]

with initial conditions

\[ x(t_0) = x_0 \quad \text{and} \quad x'(t_0) = x_1 \]

and, more generally, to any equation of the type

\[ x^{(n)}(t) = f \left( t, x(t), x'(t), \ldots, x^{(n-1)}(t) \right) \]

with initial conditions

\[ x(t_0) = x_0, x'(t_0) = x_1, \ldots, x^{(n-1)}(t_0) = x_{n-1}. \]
Exercise: Consider the equation

\[ x''(t) = x(t) - x'(t) + 2t \]

and the initial conditions

\[ x(0) = 1 \quad \text{and} \quad x'(0) = -1. \]

Estimate \( x(0.3) \) using Euler’s method with three steps.

Exercise: Consider the general equation

\[ x''(t) = f(t, x(t), x'(t)). \]

Argue, why it is reasonable to expect that the general solution to the equation will have two parameters ("two degrees of freedom").
Exercise: Write the equation

\[ x'''(t) = tx''(t) + t^2x'(t) - e^tx(t) + \cos t \]

as a system of first-order equations.

We had an example of systems of equations: “rabbits and wolves” from the first lecture:

\[
\begin{align*}
\frac{d}{dt}r(t) &= ar^2(t) + bw(t) \\
\frac{d}{dt}w(t) &= cr(t) + dw(t)
\end{align*}
\]

This is also an example of an autonomous system of two first-order equations.

\[
\begin{align*}
\frac{d}{dt}x(t) &= f(x(t), y(t)) \\
\frac{d}{dt}y(t) &= g(x(t), y(t))
\end{align*}
\]
For such systems, we can draw a direction field in the \((x, y)\)-plane. Namely, the arrow at a point \((x, y)\) points in the direction \((f(x, y), g(x, y))\).

The following computer-generated picture shows the direction field for the system

\[
\begin{align*}
r'(t) &= r^2(t) - 3w(t) \\
w'(t) &= r(t) - 2w(t)
\end{align*}
\]
Exercise: What seems to be happening to wolves and rabbits in the long run, if a) $r(0) = 2, w(0) = 0.5$, b) $r(0) = 2, w = 2$. 

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Math 216: Lecture 8