Text: Section 3.1, Practice problems # 13, 15, 19, 23, 25
For some time, we will be interested in systems of equations of the type

\[
\begin{align*}
    x'(t) & = ax(t) + by(t) \\
    y'(t) & = cx(t) + dy(t),
\end{align*}
\]  

(1)

where \(a, b, c\) and \(d\) are real numbers. The reason we care about such systems is twofold: 1) more complicated autonomous systems can be “approximated” by systems like (1) and 2) we completely understand the behavior of solutions to systems (1).

First, are some exercises that explore the linearity of more general systems of the type

\[
\begin{align*}
    x'(t) & = a(t)x(t) + b(t)y(t) \\
    y'(t) & = c(t)x(t) + d(t)y(t),
\end{align*}
\]  

(2)

Alexander Barvinok
Exercise: Suppose that $x(t)$ and $y(t)$ are solutions to the system (2), and let $\alpha$ be a real number. Show that the functions $\alpha x(t)$ and $\alpha y(t)$ are also solutions to (2).

Exercise: Suppose that $x_1(t), y_1(t)$ are solutions to the system (2) and that $x_2(t), y_2(t)$ are also solutions to that system. Show that $x(t) = x_1(t) + x_2(t)$ and $y(t) = y_1(t) + y_2(t)$ are solutions to (2).

Exercise: Suppose that $x_1(t), y_1(t)$ and $x_2(t), y_2(t)$ are solutions to the system

\[
\begin{align*}
x'(t) & = a(t)x(t) + b(t)y(t) + e(t) \\
y'(t) & = c(t)x(t) + d(t)y(t) + f(t).
\end{align*}
\]

Show that the functions $x(t) = x_1(t) - x_2(t)$ and $y(t) = y_1(t) - y_2(t)$ are solutions to (2).
We are back to systems (1). Here is some intuition. When we deal with a simple equation

$$x'(t) = ax(t),$$

the function $x(t) = e^{at}$ will be a solution (Exercise: check). For the system (1), let us try

$$x(t) = x_0 e^{\lambda t},$$
$$y(t) = y_0 e^{\lambda t},$$

where $x_0, y_0$ and $\lambda$ are some numbers. Clearly, $x_0 = y_0 = 0$ will work with any $\lambda$ (Exercise: why?), but we are interested in the case when $x_0 \neq 0$ or $y_0 \neq 0$. Substituting (3) into (1), we get
Systems of linear equations with constant coefficients

\[ x_0 \lambda e^{\lambda t} = ax_0 e^{\lambda t} + by_0 e^{\lambda t} \]
\[ y_0 \lambda e^{\lambda t} = cx_0 e^{\lambda t} + dy_0 e^{\lambda t} \]

Thus we want numbers \( x_0 \) and \( y_0 \), not both equal 0, such that

\[ ax_0 + by_0 = \lambda x_0 \]
\[ cx_0 + dy_0 = \lambda y_0 \]

for some number \( \lambda \). In this case, we say that the vector

\[
\begin{bmatrix}
  x_0 \\
  y_0
\end{bmatrix}
\]

is an eigenvector of the matrix

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\]

with eigenvalue \( \lambda \).
A bit of linear algebra

A $2 \times 2$ array

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is called a $2 \times 2$ matrix with entries $a, b, c$ and $d$. Sometimes, we index the entries of a matrix $A$ as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$ 

A $2 \times 1$ array

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

is called a 2-vector with coordinates $x$ and $y$. Sometimes, we index the coordinates of a vector $\vec{x}$ as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
A bit of linear algebra

The vector

\[ \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

is called the zero vector. We can add vectors

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}
\]

and multiply a vector by a number

\[ \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}. \]

We define the product of a $2 \times 2$ matrix $A$ and a 2-vector $\vec{x}$ by

\[
A\vec{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix},
\]

so $A\vec{x}$ is again a 2-vector.
A bit of linear algebra

The $2 \times 2$ identity matrix is the matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

All these notions extend in a natural way to matrices and vectors of larger size.

Exercise: Check that

$$A(\alpha \vec{x}) = \alpha (A\vec{x}) \quad \text{and} \quad A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}.$$ 

Exercise: Compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$ 

Exercise: Check that

$$I \vec{x} = \vec{x}$$

for all vectors $\vec{x}$. 
Here is an important definition, we were building towards.

**Definition**

Let $A$ be an $n \times n$ matrix. An $n$-vector $\vec{x}$ is called an *eigenvector* of $A$ with *eigenvalue* $\lambda$ if $\vec{x} \neq \vec{0}$ and $A\vec{x} = \lambda \vec{x}$.

**Exercise:** Show that if $\vec{x}$ is an eigenvector of $A$ with eigenvalue $\lambda$ then for any $\alpha \neq 0$ the vector $\alpha \vec{x}$ is also an eigenvector of $A$ with eigenvalue $\lambda$.

Suppose that $\vec{x}$ and $\vec{y}$ are eigenvectors of $A$ with the same eigenvalue $\lambda$ and let $\vec{z} = \vec{x} + \vec{y}$. Show that if $\vec{z} \neq \vec{0}$ then $\vec{z}$ is an eigenvector of $A$ with eigenvalue $\lambda$. 

For a $2 \times 2$ matrix, we define its determinant as
\[
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.
\]

Here are some important observations.
We have $\det A = 0$ if and only if one of the vectors
\[
\begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ d \end{bmatrix}
\]
is a scalar multiple of the other.

Exercise: Why?
In particular, if \( \det A \neq 0 \) then the lines

\[
ax + by = e \quad \text{and} \quad cx + dy = f
\]

are not parallel for any \( e \) and \( f \) and hence the system of linear equations

\[
\begin{align*}
ax + by &= e \\
\phantom{ax} + dy &= f
\end{align*}
\]

has a unique solution.
In other words, if $\det A \neq 0$ then for any vector $\vec{v}$, there is a unique solution $\vec{x}$ to the equation $A\vec{x} = \vec{v}$.

If $\det A = 0$ and if

$$
\begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ d \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

then we get parallel or coinciding lines, and hence the system

\[
\begin{align*}
ax + by &= e \\
(cx + dy) &= f
\end{align*}
\]

has either no or infinitely many solutions.
The same conclusion holds if
\[
\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Exercise: Why?
Thus if \( \det A = 0 \) then for any vector \( \vec{v} \), there are either no or infinitely many solutions \( \vec{x} \) to the equation \( A\vec{x} = \vec{v} \).
Finding eigenvalues

If $\vec{x}$ is an eigenvector of a matrix $A$ then $\vec{x} \neq \vec{0}$ with eigenvalue $\lambda$ and

$$A\vec{x} = \lambda \vec{x} \implies A\vec{x} - \lambda \vec{x} = \vec{0} \implies (A - \lambda I)\vec{x} = \vec{0}$$

Since $\vec{x} \neq \vec{0}$, the equation $(A - \lambda I)\vec{x} = \vec{0}$ has infinitely many solutions and we must have

$$\det(A - \lambda I) = 0.$$ 

Thus if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we get the equation

$$0 = \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = (a-\lambda)(d-\lambda)-bc = \lambda^2 - \lambda(a+d) + (ad-bc).$$
Finding eigenvalues

Example

Let

\[ A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \]

We find the eigenvalues by solving the equation

\[
0 = \det \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3,
\]

from which the eigenvalues are \( \lambda = 1 \) and \( \lambda = 3 \).

Here is a useful shortcut. The trace of a square matrix is defined as the sum of the diagonal entries, so that

\[
\text{trace} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d.
\]
The equation we get for the eigenvalues is

\[ 0 = \lambda^2 - \lambda(a + d) + (ad - bc), \]

that is,

\[ \lambda^2 - \text{trace}(A)\lambda + \det A = 0. \]

By Vieta’s Theorem (ater François Viéte, 1540–1603), the sum of the eigenvalues is the trace, while the product is the determinant of the matrix.

**Exercise:** Find the eigenvalues of the matrices

\[
\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 6 & -1 \\ 7 & -2 \end{bmatrix}.
\]
Finding eigenvectors

After we found the eigenvalues $\lambda$, we find eigenvectors by solving the equation $(A - \lambda I)\vec{x} = \vec{0}$ with $\vec{x} \neq \vec{0}$.

Example

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$ 

The eigenvalue, as we computed, are $\lambda = 1$ and $\lambda = 3$. We find the eigenvectors for eigenvalue $\lambda = 1$ by solving the equation

$$\begin{bmatrix} 2 - 1 & 1 \\ 1 & 2 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

that is,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
Example (Continued)

that is,

\[
x_1 + x_2 = 0 \\
x_1 + x_2 = 0.
\]

Hence

\[
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
\]

is an eigenvector with eigenvalue 1 and so is every vector

\[
a \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ -a \end{bmatrix}
\]

for any \( a \neq 0 \).
We find the eigenvectors for eigenvalue $\lambda = 3$ by solving the equation

\[
\begin{bmatrix}
2 - 3 & 1 \\
1 & 2 - 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

that is,

\[
\begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

that is,

\[
-x_1 + x_2 = 0
\]

\[
x_1 - x_2 = 0.
\]
Finding eigenvectors

Example (Finished)

Hence

\[
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

is an eigenvector with eigenvalue 3 and so is every vector

\[
a \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
a \\
a
\end{bmatrix}
\]

for any \( a \neq 0 \).

Exercise: Find the eigenvectors of the matrices

\[
\begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}
\text{ and }\begin{bmatrix}
6 & -1 \\
7 & -2
\end{bmatrix}.
\]