Instructions: Solve each of these problems. Your solution should be complete and written out in complete sentences. Where graphs are needed, you may include a print-out of output from Matlab (or another program, if you prefer).

1. In lab, we consider the Gompertz equation, \( y' = ry \ln(K/y) \) and simplifications of that using the Taylor expansion for \( \ln(y) \) near \( y = K \). Here we consider the Gompertz equation as well as the linear and quadratic approximations to the Gompertz equation we obtained in lab.

(a) Find the critical points of all three equations and determine their stability.

Solution note: For the Gompertz equation, we have the critical point \( y = K \). The value \( y = 0 \) is a tempting choice for a critical point, but as the differential equation is not defined there the model breaks down at that point and so it’s not a well-defined solution. For \( 0 < y < K \), \( y > 0 \) and \( \ln(K/y) > 0 \), so \( y' > 0 \); for \( y > K \) \( \ln(K/y) < 0 \), so \( y' < 0 \). Thus \( y = K \) is asymptotically stable.

For the linear approximation, \( y' = -r(y - K) \), the only critical point is \( y = K \). With \( f(y) = -r(y - K) \), \( f'(K) < 0 \), so this is an asymptotically stable critical point (we can also see this by noting that \( y' > 0 \) when \( y < K \) and \( y' < 0 \) when \( y > K \)).

For the quadratic approximation, \( y' = -r(y-K)-\frac{r}{2K}(y-K)^2 = \frac{r}{2K}(y-K)(y+K) \). Critical points are \( y = -K \) and \( y = K \). The right-hand side of the differential equation is a downward opening parabola, so \( y = -K \) is unstable and \( y = K \) is asymptotically stable.

(b) Solve these three equations exactly.

Solution note: For the Gompertz equation, separating variables, we have \( \frac{y'}{y(y(K)-\ln(y))} = r \), so that \( -\ln(\ln(K) - \ln(y)) = rt + C' \). Exponentiating both sides, \( \ln(K) - \ln(y) = Ce^{-rt} \) (where \( C = e^{-C'} \)). We can then solve for \( y \): \( \ln(y) = \ln(K) - Ce^{-rt} \), so that \( y = Ke^{-Ce^{-rt}} = Ke^{-Ce^{-rt}} \).

For the linear approximation we see that \( y = K + Ce^{-rt} \).
For the quadratic approximation, we again separate variables. We have \( y' = -\frac{r}{2K} \). To integrate the left-hand side, we use partial fractions: letting \( \frac{1}{(y-K)(y+K)} = \frac{A}{y-K} + \frac{B}{y+K} \), we have after clearing denominators \( 1 = A(y+K) + B(y-K) \). Thus, letting \( y = -K \) and \( y = K \), we have \( A = \frac{1}{2K} \) and \( B = -\frac{1}{2K} \).

With the partial fractions decomposition we can integrate both sides of the equation to find

\[
\frac{1}{2K} (\ln |y - K| - \ln |y + K|) = -\frac{r}{2K} t + C''.
\]

Multiplying through by \( 2K \) and letting \( C' = 2KC'' \), we have

\[
\ln \left| \frac{y - K}{y + K} \right| = -rt + C',
\]

so that, with \( C = \pm e^{C'} \), \( \frac{y - K}{y + K} = Ce^{-rt} \). Solving for \( y \),

\[
y = \frac{K(1 + Ce^{-rt})}{1 - Ce^{-rt}}.
\]

(c) Draw a phase line for each of the equations. For \( y > 0 \), is there any difference between the phase lines? Explain. Where does the difference in the solutions you obtained in (b) manifest itself?

**Solution note:** For all three we have the same phase line for \( y > 0 \),

\[
\kappa \quad y
\]

The only difference is in the time evolution of the behavior, which is the difference between the solutions we found in (b). (Though we can see from each of the equations that as \( t \to \infty \), \( y \to K \), confirming the result in the phase line.)

2. Problem 4 in §2.3 of Brannan and Boyce (p.65 in the 3rd ed. text).

**Solution note:** The modeling equation is, with \( a(t) \) as the amount of salt in the tank, \( a' = (\text{rate in}) - (\text{rate out}) \). The rate in is constant, rate in = (3 gal/min)(1 lb/gal). The rate out is dependent on the concentration of salt in the tank, which depends on the volume of water in the tank. Note that each minute we are adding 1 gal of water to the tank, so \( V(t) = 200 + t \). Then rate out = (2 gal/min)(concentration) = (2 gal/min)(\( \frac{a(t)}{V(t)} \)) = (2 gal/min)(\( \frac{a}{200+t} \) lb/gal). We are therefore solving the initial value problem.
This is linear, with integrating factor \( \mu = \exp\left(\int \frac{2}{t+200} \, dt\right) = e^{2\ln(t+200)} = (t + 200)^2 \). Thus, multiplying by \( \mu \),

\[
(a \cdot (t + 200)^2)' = 3(t + 200)^2,
\]

so that \( a \cdot (t + 200)^2 = (t + 200)^3 + C \), and \( a = (t + 200) + C(t + 200)^{-2} \). With the initial condition \( a(0) = 100 \), \( 100 = 200 + C/200^2 \), so that \( C = -100(200)^2 = -4 \times 10^6 \). Thus

\[
a(t) = (t + 200) - \frac{4 \times 10^6}{(t + 200)^2}.
\]

The tank will overflow at \( t = 300 \), when \( v = 500 \), at which point \( a(300) = 500 - \frac{4 \times 10^6}{500^2} = 484 \) lb, so that the concentration is \( \frac{484}{500} = \frac{121}{125} \approx 0.97 \). If the tank had infinite capacity, as \( t \to \infty \), \( a \to \infty \) and \( V \to \infty \). The concentration is, however \( c = \frac{a(t)}{V(t)} \), so

\[
c = \frac{(t + 200) - \frac{4 \times 10^6}{(t + 200)^2}}{t + 200} \to 1
\]
as \( t \to \infty \). That is, eventually the tank is all at the concentration of the input, which is what we would expect.

3. Problem 26 in §2.3 of Brannan and Boyce (p.68 in the 3rd ed. text). Also complete part (d), below.

**Solution note:** (a) Our equation of motion for the sled is \( v' = -\mu v^2 \), with initial condition \( v(0) = 160 \) mi/h. With \( v' = v \frac{dv}{dx} \), we have \( \frac{dv}{dx} = -\mu v \).

(b) Solving, we have \( v(x) = Ce^{-\mu x} \). Then \( v(0) = 160 \), so that \( C = 160 \) mi/h. If it takes 2200 ft = \( \frac{5}{12} \) mi to slow to 16 mi/h, we have \( v\left(\frac{5}{12}\right) = 160e^{-5\mu/12} = 16 \), so that \( \mu = \frac{12}{5} \ln(10) \approx 5.526 \).

(c) To find the time it takes to slow to 16 mi/h, we need \( v(t) \). We can find this by solving the original differential equation by separating variables. We have \( \frac{v'}{v^2} = -\mu \), so that \( -\frac{1}{v} = -\mu t + C \), and \( v(t) = \frac{1}{C-\mu t} \). \( v(0) = 160 \), so \( C = \frac{1}{160} \), and \( v(t) = \frac{160}{1 + 160 \mu t} \). We have \( v(t) = 16 = \frac{160}{t + 160 \mu t} \), so that \( t = \frac{9}{160 \mu} \approx 0.01 \) hr, or about 37 sec.

(d) What does this model say about the time it takes for the sled to come to a complete stop?

**Solution note:** We see that the sled comes to a complete stop only as \( t \to \infty \): that is, the model predicts that it never fully stops.
4. Problem 11 in §2.4 of Brannan and Boyce (p.79 in the 3rd ed. text). Also complete parts (a)–(d), below.

Solution note: We see that \( y' = f(t,y) = \frac{1+t^2}{3y^3-y} \), so that \( f \) and \( f_y \) are discontinuous at \( y = 0 \) and \( y = 3 \). The hypotheses of the theorem are therefore satisfied for all \( t \), and \( y \neq 0,3 \).

(a) Solve the equation with the initial condition \( y(0) = 1 \) (you will be able to find an implicit equation for \( y \)).

Solution note: Separating variables, we have \( \frac{3}{2}y^2 - \frac{1}{3}y^3 = t + \frac{1}{3}t^3 + C \). With \( y(0) = 1 \), \( C = \frac{7}{6} \), so that \( \frac{3}{2}y^2 - \frac{1}{3}y^3 = t + \frac{1}{3}t^3 + \frac{7}{6} \).

(b) Based on your answer to problem 11, find the range of \( t \) and \( y \) values on which you would expect the solution to exist. (Consider \( t \) values both greater and less than 0; to find exact values you will need to solve the equation you obtain numerically.)

Solution note: We look for \( t \) where \( y = 0 \) or \( y = 3 \). If \( y = 0 \), \( 0 = \frac{1}{3}t^3 + t + \frac{7}{6} \). This isn’t easy to solve by hand, but numerically we find \( t = -0.913 \). Similarly, if \( y = 3 \), we have \( t = 1.699 \).

(c) Use Matlab or some other tool to draw the direction field for the equation on an appropriate domain. Sketch the solution you found in (a) on the direction field.

Solution note:

(d) Solve the initial value problem \( y' = \frac{1+t^2}{3y^3-y} \), \( y(0) = 1 \) with Matlab using ode45, and show that the numerical solution breaks down at the times you found in (b). (You will want to pick the range on which you solve the initial value problem carefully; then plot your}

\footnote{e.g., <https://www.geogebra.org/m/W7dAdgqc>
solution.) Publish your solution as you did lab 0, and include it with your homework.

**Solution note:**

We are solving 

\[
\begin{align*}
\frac{dy}{dt} &= \frac{1 + t^2}{3y - y^2}, \\
\end{align*}
\]

, to see where it breaks down.

\[
[t1,y1] = \text{ode45}( @(t,y) \frac{1 + t^2}{3y - y^2}, [0 1.8], 1 );
\]

\[
[t2,y2] = \text{ode45}( @(t,y) \frac{1 + t^2}{3y - y^2}, [0 -0.915], 1 );
\]

We note that the solution has clearly broken down at the right end point. Trying to push the left endpoint further to the left results in the calculation in Matlab running seemingly interminably.