Math 216–W20 Written Homework 2 Solutions

Instructions: Solve each of these problems. Your solution should be complete and written out in complete sentences.

1. Problem 22 in §3.1 of Brannan and Boyce (p.129 in the 3rd ed. text). Also complete (a)–(c), below.

\[ \text{Solution note: We find eigenvalues by requiring that } \det(A - \lambda I) = \det\begin{pmatrix} 6 - \lambda & 3 \\ 2 & 1 - \lambda \end{pmatrix} = (6 - \lambda)(1 - \lambda) - 6 = \lambda^2 - 7\lambda = 0. \text{ Thus } \lambda = 0 \text{ or } \lambda = 7. \text{ If } \lambda = 0, \text{ the eigenvector satisfies } Av = 0, \text{ or, with } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, 6v_1 + 3v_2 = 2v_1 + v_2 = 0, \text{ so we may take } v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \text{ If } \lambda = 7, \text{ the eigenvector satisfies } \begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix} v = 0, \text{ so that } v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \]

(a) Without doing any further work, how many solutions are there to \[ A x = 0 \] ? What are they?

\[ \text{Solution note: This is exactly the equation for the eigenvector corresponding to } \lambda = 0, \text{ so we know that there are an infinite number of solutions } x = k \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \]

(b) Similarly without doing any further work, how many solutions are there to \[ \begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix} x = 0? \]

\[ \text{Solution note: Similarly, this is exactly the equation for the eigenvector corresponding to } \lambda = 7, \text{ so we know that there are an infinite number of solutions } x = k \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \]

(c) Without doing any further work, how many solutions are there to \[ A x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}? \text{ To } A x = \begin{pmatrix} 3 \\ 1 \end{pmatrix}? \text{ Solve each equation to confirm your conclusion.} \]

\[ \text{Solution note: Because } \det A = 0, \text{ we know that both of these have either no solution or an infinite number. The first row of } A \text{ is three times the second, so we expect that there will be an infinite number of solutions to the second system (where the right-hand side has the same characteristic) and none for the first.} \]
Solving the first explicitly, we need \(6x_1 + 3x_2 = 1\) and \(2x_1 + x_2 = 1\). The second tells us that \(x_2 = 1 - 2x_1\), so, substituting into the first, we have \(6x_1 + 3 - 6x_1 = 1\), or \(3 = 1\), which is false. So there are no solutions, as we expected.

For the second equation, we’re solving \(6x_1 + 3x_2 = 3\) and \(2x_1 + x_2 = 1\). Dividing the first by 3, we see that both equations require only that \(2x_1 + x_2 = 1\). Thus we can pick any value for \(x_1\), and \(x_2 = 1 - 2x_1\); the solution is therefore \(\mathbf{x} = \begin{pmatrix} x_1 \\ 1 - 2x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}\). Note that the second vector is the eigenvector corresponding to \(\lambda = 0\): that is, we’re writing the solution as \(\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (\text{any multiple of}) (\text{the solution to } \mathbf{A}\mathbf{v} = \mathbf{0})\).

2. Problem 26 in §3.3 of Brannan and Boyce (p.166 in the 3rd ed. text).

**Solution note:**

(a) With \(x_1 = y\), \(x_2 = y'\), we have \(\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 0 & \frac{4}{5} \end{pmatrix} \mathbf{x}\).

(b) Because it is triangular, eigenvalues of the coefficient matrix are the diagonal entries, \(\lambda = 0\) and \(\lambda = \frac{4}{5}\). If \(\lambda = 0\), \(\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\), and if \(\lambda = \frac{4}{5}\), \(\mathbf{v} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}\). Thus the general solution to the system is \(\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 5 \\ 4 \end{pmatrix} e^{4t/5}\).

(c) Then \(y = c_1 + c_2 e^{4t/5}\).

(d) A phase portrait is shown below.

(e) Critical points are \(x_1 = \text{any value}, x_2 = 0\), so there is a line of critical points along the \(x_1\)-axis. Because everywhere off the axis trajectories move away to infinity, these are all unstable.

3. In lab 2 we consider the van der Pol equation, \(x'' + \mu(x^2 - 1)x' + x = 0\).
(a) Show that at the critical point $x = 0$, the linearized system is that given in the lab, $x' = y$, $y' = -x + \mu y$.

**Solution note:** To linearize at $x = 0$, we let $x = 0 + u$, with $|u|$ very small. Then $u'' + \mu(u^2 - 1)u' + u = 0$, or, taking advantage of $|u|$ being small (so that $u^2 u'$ is likely very, very small), $u'' + \mu u' + u = 0$. Rewriting this as a system with $x = u$ and $y = x' = u'$, we have $x' = y$, $y' = -x + \mu y$.

(b) Solve the system for $\mu = 0$, $\mu = 0.1$, and $\mu = 3$.

**Solution note:** As a system, this is $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Note that for any $\mu$, the first row of the equation for the corresponding eigenvector $v$ will give $v = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$.

If $\mu = 0$, eigenvalues satisfy $\lambda^2 + 1 = 0$, so $\lambda = \pm i$, and given the eigenvectors above, we can write the solution $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$.

If $\mu = 0.1$, $\lambda^2 - 0.1\lambda + 1 = 0$, so $\lambda = 0.05 \pm i \cdot 0.5\sqrt{3.99}$. Then the general solution is $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \cos(\sqrt{3.99} t) \\ 0.05\cos(\sqrt{3.99} t) - 0.5\sqrt{3.99}\sin(\sqrt{3.99} t) \end{pmatrix} e^{0.05t} + c_2 \begin{pmatrix} \sin(\sqrt{3.99} t) \\ 0.5\sqrt{3.99}\cos(\sqrt{3.99} t) + 0.05\sin(\sqrt{3.99} t) \end{pmatrix} e^{0.05t}$.

Finally, if $\mu = 3$, $\lambda^2 - 3\lambda + 1 = 0$, so that $\lambda = \frac{3}{2} \pm \sqrt{5}$. The general solution is then $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ (3 - \sqrt{5})/2 \end{pmatrix} e^{(3-\sqrt{5})t/2} + c_2 \begin{pmatrix} 1 \\ (3 + \sqrt{5})/2 \end{pmatrix} e^{(3+\sqrt{5})t/2}$.

(c) For each of these, sketch, by hand, the phase portraits for the system. Be careful to note how they differ.

**Solution note:** Note that for any $\mu$, at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, so the trajectory moves downward, and around a center or spiral point trajectories will rotate clockwise.

Then, for $\mu = 0$ we have a center, $\mu = 0.1$ a spiral, and $\mu = 3$ a node, as shown in the figures below.
4. Problem 12 in §3.5 of Brannan and Boyce (p.188 in the 3rd ed. text).
Change the sign of the diagonal entries in the matrix, however, so that the coefficient matrix is \( A = \begin{pmatrix} -2 & \frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix} \). To work the problem, complete (a)--(e), below.

(a) Find the general solution to the system.

\[
\text{Solution note: We first find eigenvalues and eigenvectors. The eigenvalues satisfy } \det(A - \lambda I) = 0, \text{ or } (-2 - \lambda)(-1 - \lambda) + \frac{1}{4} = \lambda^2 + 3\lambda + \frac{9}{4} = (\lambda + \frac{3}{2})^2 = 0, \text{ so we have the repeated eigenvalue } \lambda = -\frac{3}{2}. \text{ With this eigenvalue, the eigenvector(s) satisfies (y)}
\]

\[-\frac{1}{2}v_1 + \frac{1}{2}v_2 = 0, \text{ so } v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \text{ To write the general solution to the problem, we need the generalized eigenvector } w \text{ solving } (A - \lambda v)w = v, \text{ which is } \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \text{ A solution } w \text{ is } w = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \text{ The general solution is therefore}
\]

\[
x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \left( \frac{1}{1} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) e^{-t/2}.
\]

(b) Find the solution for the given initial condition.
Solution note: Applying the initial condition, we have $c_1 = 1$ and $c_2 = 1$, so

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t/2}.$$

(c) Sketch, by hand, the phase portrait for the system.

Solution note: To sketch phase portraits, we first draw the straight line solution along $y = x$, that has trajectories converging to the origin. Then note that a solution that starts above the origin will have $c_1 = 0$ and a nonzero value of $c_2$. As $t$ increases, the term $\begin{pmatrix} 1 \\ 1 \end{pmatrix} t$ will push the trajectory in the positive $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ direction initially, after which it will decay to the origin. This gives the phase portrait shown below.

(d) Sketch the trajectory corresponding to your solution in (b) in the phase plane, and use this to sketch the component plots.

Solution note: The dashed curve in the figure above is the trajectory for the indicated initial condition. We see that as time increases, $x$ increases from 1 and then decreases to zero, and $y$ starts at 3, may increase, and then decreases to zero. This gives the sketch shown below.

(e) Use Matlab to plot the component plots given your solution in (b). Note how this is similar or different from what you obtained in (d).
Solution note: Our solution is \( x = e^{-t/2} + te^{-t/2} \), \( y = 3e^{-t/2} + te^{-t/2} \). We can plot this in Matlab with the following commands:

\[
\begin{align*}
\text{>> } & t = 0:.01:10; \\
\text{>> } & x = \exp(-t/2) + t.*\exp(-t/2); \\
\text{>> } & y = 3*\exp(-t/2) + t.*\exp(-t/2); \\
\text{>> } & \text{plot( } t,x,'-k', t,y,'--k', 'LineWidth', 2 ); \\
\text{>> } & \text{xlabel('t'); ylabel('x,y');}
\end{align*}
\]

This clarifies that \( y \) does not increase before decreasing to zero, and gives the horizontal scale for the decay (from the phase portrait we are unable to determine the time scale).