Written Homework 1

1. Consider the initial value problem

\[ x' = -tx + x^3, \quad x(0) = \alpha, \]

where \( \alpha \) is a parameter.

(a) Prove that for \( \alpha = 1 \), the solution \( x(t) \) grows to infinity (explodes) somewhere on the interval \((0,1)\) completing the following steps

i. Find the solution of initial value problem \( y' = y^2, \quad y(0) = 1 \). We denote it \( y(t) \) and use in the next steps.

ii. Compute \( x'(0), x''(0), x'''(0), y'(0), y''(0), y'''(0) \) using differential equations.

iii. Conclude that for small \( t > 0 \) we have \( x(t) > y(t) \) using Taylor formula.

iv. Show that \( y(t) > \sqrt{t} \) on the interval \( 0 < t < 1 \).

v. Show that on the interval \( 0 < t < 1 \) if \( x(t) > y(t) \), then

\[ x(t)(x^2(t) - t) > y(t)(y^2(t) - t). \]

vi. Show that \( y(t)(y^2(t) - t) > y^2(t) \) on the interval \( 0 < t < 1 \).

vii. Conclude that while \( x(t) \) is defined, it is greater than \( y(t) \). Therefore it explodes to infinity at \( t = 1 \) or earlier.

(b) Use Matlab to draw a direction field for the differential equation (and attach your graph to your solutions). Observe that there is a critical value of \( \alpha \) in the interval \( 0 \leq \alpha \leq 1 \) such that if \( \alpha < \alpha_0 \), then \( x(t) \) converges to zero and if \( \alpha \geq \alpha_0 \), then \( x(t) \) explodes to infinity. Call this critical value \( \alpha_0 \).

(c) Plot the solutions for various \( \alpha \). Estimate \( \alpha_0 \) by restricting it to an interval \([a,b]\), where \( b - a = 0.1 \).

2. In Lab 1, you use the Gompertz equation to model the growth of cancer cells in a tumor:

\[ \frac{dy}{dt} = ry \ln(K/y), \]

where \( r, K > 0 \) are constants. You also consider the quadratic approximation of the Gompertz equation

\[ \frac{dy}{dt} = -r(y - K) - \frac{r}{2K}(y - K)^2. \]

(a) For each equation, sketch a graph of the right-hand side function (i.e. Gompertz: \( f(y) = ry \ln(K/y) \), Approximation: \( f(y) = -r(y - K) - \frac{r}{2K}(y - K)^2 \), versus \( y \). Find the equilibrium solutions and determine their stability.
(b) Find the general solution to each equation.
(c) Given the initial condition \( y(0) = \frac{K}{2} \), what is the long-term behavior of the solution to each equation?

3. Consider the family of curves on the plane described by implicit equation \( F(x, y) = a \) where \( a \) is a parameter. By taking derivative with respect to \( x \) and using chain rule we obtain the differential equation \( \frac{dy}{dx} = -\frac{\partial F}{\partial x} \frac{\partial F}{\partial y} \). The family of orthogonal curves is then described by equation \( \frac{dy}{dx} = \frac{\partial F}{\partial y} \frac{\partial F}{\partial x} \). Construct the family of curves orthogonal to the family \( \sin(y) = a \cos(x) \), and illustrate your solution with plots.

4. A mass of 0.4 kg is dropped from rest in a medium offering a resistance of 0.2\( |v| \), where \( v \) is measured in meters per second. Acceleration due to gravity is 9.8 m/s\(^2\).

(a) If the mass is dropped from a height of 25 m, find its velocity when it hits the ground.
(b) If the mass is to attain a velocity of no more than 8 m/s, find the maximum height from which it can be dropped.
(c) Suppose that the resistive force is \( k|v| \), where \( v \) is measured in m/s, and \( k \) is constant. If the mass is dropped from a height of 25 m and must hit the ground with a velocity of no more than 8 m/s, determine the coefficient of resistance \( k \) that is required.