Math 216 — Second Midterm
1 April, 2020

1. This exam is to be completed with pencil-and-paper, and a scan or image of your work submitted on-line. Your exam is not complete until you have submitted your work there.

2. There are 5 problems on this exam. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.

3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions.

4. Show all of your work, including explanation for what you are doing and why for each problem, so that graders can see not only your answer but how you obtained it. Because this is an online test, work submitted with no explanation may be given no credit.

5. You may use no aids (e.g., calculators or notecards) on this exam.

6. Turn off all cell phones, remove all headphones, and place any watch you are using on the table or desk in front of you.

7. This is the hardcopy of the second midterm. You should work each problem here, scan or make an image of your work, and submit the work for each problem in the Canvas quiz where you are submitting work for the exam. You do not need to do your work on a printout of this exam; it may be done on any plain paper. Your submitted work is essential for credit. If you have trouble submitting your work please be sure to contact your instructor.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>75</td>
<td></td>
</tr>
</tbody>
</table>
1. [15 points] Consider the equation $y'' + (2 - k)y' + (1 - k)y = 0$. Find all values of the constant $k$ for which solutions to the equation may become unbounded as $t \to \infty$. Give an initial condition for which the solution is unbounded.

Solution: This is a homogeneous problem, so we expect that solutions will be of the form $y = e^{\lambda t}$. Plugging in, $\lambda$ satisfies the characteristic equation,

$$\lambda^2 + (2 - k)\lambda + (1 - k) = 0,$$

so that

$$\lambda = \frac{1}{2}(k - 2) \pm \frac{1}{2}\sqrt{(k - 2)^2 - 4(1 - k)} = \frac{1}{2}(k - 2) \pm \frac{1}{2}\sqrt{k^2 - 4k + 4 - 4 + 4k} = \frac{1}{2}(k - 2) \pm \frac{1}{2}k.$$ 

Thus $\lambda = k - 1$ or $\lambda = -1$. If $c \neq 0$, the general solution is $y = c_1e^{(k-1)t} + c_2e^{-t}$, and if $k = 0$, it is $y = c_1te^{-t} + c_2e^{-t}$. Thus, for all $c \leq 1$ solutions remain bounded as $t \to \infty$, and we require $c > 1$ for any solution to be unbounded. If $c > 1$, any initial condition other than $y(0) = a$, $y'(0) = -a$ (for some constant $a$) will have $c_1 \neq 0$, and thus will become unbounded.
2. [15 points] Consider an RLC circuit, such as we considered in lab 4. If the forcing voltage is constant, the system may be modeled by

\[ y'' + cy' + ky = V_0, \]

where \( c, k, \) and \( V_0 \) are positive constants; we assume that \( c \) is small relative to \( k \).

a. [7 points] Find the general solution to this equation.

**Solution:** We know the general solution will be of the form \( y = y_c + y_p \), where \( y_c \) is the solution to the complementary homogenous problem. For that, we look for solutions of the form \( y = e^{\lambda t} \). Then \( \lambda \) satisfies the characteristic equation \( \lambda^2 + c\lambda + k = 0 \), so that \( \lambda = -\frac{c}{2} \pm \frac{i\sqrt{k^2 - c^2}}{2} \). Because \( c \) is small relative to \( k \), we expect that \( c^2 - 4k < 0 \), so that 

\[ \lambda = -\frac{c}{2} \pm \frac{i\omega_0}{2}, \]

and 

\[ y_c = c_1 e^{-ct/2} \cos(\omega_0 t) + c_2 e^{-ct/2} \sin(\omega_0 t), \]

with \( \omega_0 = \sqrt{k - \left(\frac{c}{2}\right)^2} \).

b. [8 points] Now suppose that we introduce an oscillatory forcing voltage, so that

\[ y'' + cy' + ky = V_0 \cos(\omega t). \]

For \( \omega = 0.3 \), \( \omega = 0.9 \), and \( \omega = 2 \) solutions to the model are shown to the right, below. Which solution curve corresponds to which value of \( \omega \), and why? Approximately, what do you think that the amplitude of the steady state response will be if \( \omega = 1.5 \)? \( \omega = 0.7 \)? Explain.

**Solution:** We can match the solutions to the frequencies \( \omega \) by looking at the period of the oscillations shown. The dashed curve has the longest period and hence smallest \( \omega \), \( \omega = 0.3 \). The solid curve has the next longest period, and is \( \omega = 0.9 \), and the last, dotted, curve is \( \omega = 2 \). We note that the amplitude of the oscillation increases slightly, and then decreases. This is shown in the lower figure to the right. The dashed curve shows a curve through these points, which reflects the form of a gain function that we expect in this type of situation. Given this, we expect that the amplitude when \( \omega = 1.5 \) will be about 0.5, and at \( \omega = 0.7 \) about 1.2.
3. [15 points] Suppose that we are solving a linear, second order, constant-coefficient, nonhomogeneous differential equation in $y(t)$ using Laplace transforms. If the forcing term is $f(t) = e^{-5t}$, we find $Y(s) = \mathcal{L}\{y(t)\} = -\frac{s + 1}{s^2 + 2s + 5} + \frac{1}{(s + 5)(s^2 + 2s + 5)}$.

a. [8 points] What is the equation being solved, and what are the initial conditions?

Solution: We know that the forward transform of the equation $y'' + py' + qy = e^{-5t}$ with initial conditions $y(0) = y_0$, $y'(0) = v_0$ will be

$$(s^2 + ps + q)Y(s) - y_0s - (py_0 + v_0) = \frac{1}{s + 5},$$

so that

$$Y(s) = \frac{y_0s + py_0 + v_0}{s^2 + ps + q} + \frac{1}{(s + 5)(s^2 + ps + q)}.$$

Matching this against the expression given, we see that $y_0 = -1$, $p = 2$, $q = 5$, and $v_0 = 1$. We are therefore solving

$$y'' + 2y' + 5y = e^{-5t}, y(0) = -1, y'(0) = 1.$$

b. [7 points] Using Laplace transform techniques, invert the expression for $Y(s)$ to find the solution $y(t)$ to the problem.

Solution: First note that the expression for $Y(s)$ is

$$Y(s) = -\frac{s + 1}{(s + 1)^2 + 4} + \frac{A}{s + 5} + \frac{B(s + 1) + C}{(s + 1)^2 + 4},$$

for some $A$, $B$, and $C$, and the resulting inverse transform is

$$y(t) = -e^{-t}\cos(2t) + Ae^{-5t} + Be^{-t}\cos(2t) + \frac{1}{2}Ce^{-t}\sin(2t).$$

To find $A$, $B$, and $C$, we finish the partial fractions calculation, letting

$$\frac{1}{(s + 5)((s + 1)^2 + 4)} = \frac{A}{s + 5} + \frac{B(s + 1) + C}{(s + 1)^2 + 4},$$

so that

$$1 = A((s + 1)^2 + 4) + B(s + 5)(s + 1) + C(s + 5).$$

If $s = -5$, we have $A = \frac{1}{20}$; then, matching powers of $s^2$ on either side of the equation, $B = -\frac{1}{20}$. Finally, letting $s = -1$, $C = \frac{1}{5}$. Thus $y(t) = -e^{-t}\cos(2t) + \frac{1}{20}e^{-5t} - \frac{1}{20}e^{-t}\cos(2t) + \frac{1}{10}e^{-2t}\sin(2t)$. 
4. [15 points] Write an initial value problem that could model a forced (linear) spring with solution \( y(t) \) shown in the figure to the right, below.

\[ y'' + cy' + ky = A \cos(\pi t), \quad y(0) = 0, \quad y'(0) = 0, \]

for some constants \( c > 0, k > 0, \) and \( A. \)

5. [15 points] Suppose that you are solving a nonhomogeneous, second-order, linear, constant-coefficient differential equation with forcing \( f(t) = 5e^{-2t} \). Write the initial value problem that has the solution \( y(t) = 2e^{-t} \cos(2t) + e^{-2t} \).

\[ y'' + 2y' + 5y = 5e^{-5t}. \]

Note that plugging in \( y_p = e^{-2t} \), we have \((4 - 4 + 5)e^{-2t} = 5e^{-5t}\), so this appears to be correct! Finally, the general solution to this is

\[ y = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) + e^{-2t}, \]

and from the solution we are given we therefore know that \( c_1 = 2 \) and \( c_2 = 0 \). Then \( y(0) = c_1 + 1 = 3; \) and \( y'(0) = -c_1 - 2 = -4. \)