Math 216 — Second Midterm
1 April, 2020

1. This exam is to be completed with pencil-and-paper, and a scan or image of your work submitted on-line. Your exam is not complete until you have submitted your work there.

2. There are 5 problems on this exam. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.

3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions.

4. Show all of your work, including explanation for what you are doing and why for each problem, so that graders can see not only your answer but how you obtained it. Because this is an online test, work submitted with no explanation may be given no credit.

5. You may use no aids (e.g., calculators or notecards) on this exam.

6. Turn off all cell phones, remove all headphones, and place any watch you are using on the table or desk in front of you.

7. This is the hardcopy of the second midterm. You should work each problem here, scan or make an image of your work, and submit the work for each problem in the Canvas quiz where you are submitting work for the exam. You do not need to do your work on a printout of this exam; it may be done on any plain paper. Your submitted work is essential for credit. If you have trouble submitting your work please be sure to contact your instructor.

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1. [15 points] Consider a cylindrical buoy floating in a body of water, as suggested by the figure to the right. The water exerts a buoyant force on the buoy that pushes it up, and the weight of the buoy pulls it down. If we assume that the water is still, the depth at which the buoy floats may be modeled by the equation

\[ y'' + 2\gamma y' + ky = g, \]

where \( \gamma, k, \) and \( g \) are positive constants, and \( \gamma \) is small relative to \( k \).

a. [7 points] Find the general solution to this equation.

**Solution:** We know the general solution will be of the form \( y = y_c + y_p \), where \( y_c \) is the solution to the complementary homogeneous problem. For that, we look for solutions of the form \( y = e^{\lambda t} \). Then \( \lambda \) satisfies the characteristic equation \( \lambda^2 + 2\gamma \lambda + k = 0 \), so that \( \lambda = -\gamma \pm \sqrt{\gamma^2 - k} \). Because \( \gamma \) is small relative to \( k \), we expect that \( \gamma^2 - k < 0 \), so that this is \( \lambda = -\gamma \pm i\sqrt{k - \gamma^2} = -\gamma \pm i\delta \), and \( y_c = c_1 e^{-\gamma t} \cos(\omega_0 t) + c_2 e^{-\gamma t} \sin(\omega_0 t) \). Because the forcing is a constant we look for a constant particular solution, \( y_p = a \), so that \( ka = g \), and \( a = g/k \). The general solution is therefore

\[ y = c_1 e^{-\gamma t} \cos(\omega_0 t) + c_2 e^{-\gamma t} \sin(\omega_0 t) + \frac{g}{k}, \]

with \( \omega_0 = \sqrt{k - \gamma^2} \).

b. [8 points] Suppose that the water is not still, so that the model becomes

\[ y'' + 2\gamma y' + ky = F_0 \cos(\omega t) \]

For \( \omega = 0.3, \) \( \omega = 1, \) and \( \omega = 1.5, \) the amplitude \( A \) of the steady state oscillation of the buoy is shown in the figure to the right. Approximately, what do you think that the amplitude of the steady state response will be if \( \omega = 0.7 \)? Without doing additional work to solve the equation, write the general solution to the problem in this case and sketch a graph of what you would expect \( y(t) \) to look like.

**Solution:** Note that because we don’t know what the forcing amplitude \( F_0 \) is, we can’t tell if this is demonstrating resonance or not. However, we expect that the amplitude as a function of the forcing frequency will look something like the second graph shown to the right. Therefore, we expect that the amplitude at \( \omega = 0.7 \) should be larger than that at either \( \omega = 0.3 \) or \( \omega = 1 \). A reasonable guess would be that \( A \) is between 1.1 and 1.5. Then, in this case we know that the particular solution will be of the form \( y_p = a_1 \cos(\omega t) + a_2 \sin(\omega t) = A \cos(\omega t - \delta) \). Without doing further work we cannot find \( \delta \), but taking \( A = 1.2 \) know that the general solution is

\[ y = c_1 e^{-\gamma t} \cos(\omega_0 t) + c_2 e^{-\gamma t} \sin(\omega_0 t) + 1.2 \cos(0.7t - \delta), \]

with \( \omega_0 = \sqrt{k - \gamma^2} \). The graph of this will be a transient followed by a steady oscillation with frequency \( \omega = 0.7 \) and amplitude \( A = 1.2 \), as in the last graph to the right.
2. [15 points] Suppose that you are solving a second-order linear, constant-coefficient differential equation with forcing \( f(t) = 4e^{-t} \). If the particular solution is \( y_p = 2t^2e^{-t} \), what is the differential equation and its general solution?

**Solution:** We know that we are guessing \( y_p = At^2e^{-t} \) when the forcing term is \( f(t) = 4e^{-t} \), which means that both \( e^{-t} \) and \( te^{-t} \) must be solutions to the complementary homogeneous equation. Thus the differential equation must be \( a(y'' + 2y' + y) = 4e^{-t} \), for some constant \( a \). Then, with \( y_p = 2t^2e^{-t} \), we have \( y_p = 4te^{-t} - 2t^2e^{-t} \) and \( y_p = 4e^{-t} - 8te^{-t} + 2t^2e^{-t} \). Plugging these into the differential equation, we have after cancellation \( 4ae^{-t} = 4e^{-t} \), so \( a = 1 \). Thus the differential equation is

\[
y'' + 2y' + y = 4e^{-t},
\]

and the general solution is

\[
y = c_1e^{-t} + c_2te^{-t} + 2t^2e^{-t}.
\]

3. [15 points] Consider the equation \( y'' + (2 - c)y' + (1 - c)y = 0 \). Find all values of the constant \( c \) for which all solutions to the equation remain bounded as \( t \to \infty \). If possible, find a value of \( c \) and an initial condition for \( y \) that result in a solution that remains bounded but does not go to zero.

**Solution:** This is a homogeneous problem, so we expect that solutions will be of the form \( y = e^{\lambda t} \). Plugging in, \( \lambda \) satisfies the characteristic equation,

\[
\lambda^2 + (2 - c)\lambda + (1 - c) = 0,
\]

so that

\[
\lambda = \frac{1}{2}(c - 2) \pm \frac{1}{2}\sqrt{(c - 2)^2 - 4(1 - c)}
\]

\[
= \frac{1}{2}(c - 2) \pm \frac{1}{2}\sqrt{c^2 - 4c + 4 - 4 + 4c} = \frac{1}{2}(c - 2) \pm \frac{1}{2}c.
\]

Thus \( \lambda = c - 1 \) or \( \lambda = -1 \). If \( c \neq 0 \), the general solution is \( y = c_1e^{(c-1)t} + c_2e^{-t} \), and if \( c = 0 \), it is \( y = c_1te^{-t} + c_2e^{-t} \). Thus, for all \( c \leq 1 \) solutions remain bounded as \( t \to \infty \). If \( c = 1 \), the general solution is \( y = c_1 + c_2e^{-t} \), and any initial condition other than \( y(0) = k, y'(0) = -k \) will have \( y \to c_1 \), a nonzero constant. (If \( y(0) = k \) and \( y'(0) = -k \) we must take \( c_1 = 0 \) so that the solution is \( y = ke^{-t} \), which goes to zero as \( t \to \infty \).)
4. [15 points] Suppose that we are solving a linear, second order, constant-coefficient, nonhomogeneous differential equation in \( y(t) \) using Laplace transforms. If the forcing term is \( f(t) = e^{-t} \), we find \( Y(s) = \mathcal{L}\{y(t)\} = -\frac{s + 2}{s^2 + 4s + 5} + \frac{1}{(s + 1)(s^2 + 4s + 5)} \).

a. [8 points] What is the equation being solved, and what are the initial conditions?

**Solution:** We know that the forward transform of the equation \( y'' + py' + qy = e^{-t} \) with initial conditions \( y(0) = y_0, y'(0) = v_0 \) will be

\[
(s^2 + ps + q)Y(s) - y_0s - (py_0 + v_0) = \frac{1}{s + 1},
\]

so that

\[
Y(s) = \frac{y_0s + py_0 + v_0}{s^2 + ps + q} + \frac{1}{(s + 1)(s^2 + ps + q)}.
\]

Matching this against the expression given, we see that \( y_0 = -1, p = 4, q = 5, \) and \( v_0 = 2 \). We are therefore solving

\[
y'' + 4y' + 5y = e^{-t}, y(0) = -1, y'(0) = 2.
\]

b. [7 points] Using Laplace transform techniques, invert the expression for \( Y(s) \) to find the solution \( y(t) \) to the problem.
Solution: First note that the expression for $Y(s)$ is

$$Y(s) = -\frac{s + 2}{(s + 2)^2 + 1} + \frac{A}{s + 1} + \frac{B(s + 2) + C}{(s + 2)^2 + 1},$$

for some $A$, $B$, and $C$, and the resulting inverse transform is

$$y(t) = -e^{-2t} \cos(t) + Ae^{-t} + Be^{-2t} \cos(t) + Ce^{-2t} \sin(t).$$

To find $A$, $B$, and $C$, we finish the partial fractions calculation, letting

$$\frac{1}{(s + 1)((s + 2)^2 + 1)} = \frac{A}{s + 1} + \frac{B(s + 2) + C}{(s + 2)^2 + 1},$$

so that

$$1 = A((s + 2)^2 + 1) + B(s + 1)(s + 2) + C(s + 1).$$

If $s = -1$, we have $A = \frac{1}{2}$; then, matching powers of $s^2$ on either side of the equation, $B = -\frac{1}{2}$. Finally, letting $s = -2$, $C = -\frac{1}{2}$. Thus $y(t) = -e^{-2t} \cos(t) + \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t} \cos(t) - \frac{1}{2}e^{-2t} \sin(t).

In the on-line version, this is

$$Y(s) = -\frac{s + 2}{(s + 2)^2 + 1} + \frac{A}{s + 2} + \frac{B(s + 2) + C}{(s + 2)^2 + 1},$$

for some $A$, $B$, and $C$, and the resulting inverse transform is

$$y(t) = -e^{-2t} \cos(t) + Ae^{-t} + Be^{-2t} \cos(t) + Ce^{-2t} \sin(t).$$

To find $A$, $B$, and $C$, we finish the partial fractions calculation, letting

$$\frac{1}{(s + 2)((s + 2)^2 + 1)} = \frac{A}{s + 2} + \frac{B(s + 2) + C}{(s + 2)^2 + 1},$$

so that

$$1 = A((s + 2)^2 + 1) + B(s + 2)^2 + C(s + 2).$$

If $s = -2$, we have $A = 1$; then, matching powers of $s^2$ on either side of the equation, $B = -1$. Finally, letting $s = -1$, $C = 0$. Thus $y(t) = -e^{-2t} \cos(t) + e^{-2t} - e^{-2t} \cos(t).$
5. [15 points] For what constants $p$, $q$, $y_0$, and $v_0$ could the initial value problem $y'' + py' + qy = 0$, $y(0) = y_0$, $y'(0) = v_0$ give the solution $y(t)$ shown in the figure to the right? (Note that you should be able to determine most, but possibly not all, of these constants precisely.)

**Solution:** First note that $y(0) = 1$ in the graph, and $y'(0)$ is hard to read; it might be zero. Next note that the solution is a decaying oscillation. Thus

$$y = c_1 e^{-rt} \cos(\omega t) + c_2 e^{-rt} \sin(\omega t)$$

for some $r$ and $\omega$. We see that successive peaks of the oscillatory part of the solution occur at integer values, so $\omega = \pi$ (and the period of the cosine and sine terms is 2). This tells us that the roots of the characteristic polynomial are $\lambda = -r \pm i\pi$. The characteristic polynomial is, using these roots,

$$\lambda^2 + p\lambda + q = (\lambda + r)^2 + \pi^2.$$ 

Expanding the right-hand side and matching terms, $p = 2r$ and $q = r^2 + \pi^2$. We can’t easily determine $r$ from the graph; it could be $r = 1$, in which case $p = 2$, $q = 1 + \pi^2$, $y_0 = 1$, and $v_0 = 0$. 