1. This exam is to be completed with pencil-and-paper, and a scan or image of your work submitted on-line. Your exam is not complete until you have submitted your work there.

2. There are 7 problems on this exam. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.

3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions.

4. **Show all of your work, including explanation for what you are doing and why for each problem, so that graders can see not only your answer but how you obtained it. Because this is an online test, work submitted with no explanation may be given no credit.**

5. **You may use no aids (e.g., calculators or notecards) on this exam.**

6. **Turn off all cell phones, remove all headphones, and place any watch you are using on the table or desk in front of you.**

7. **This is the hardcopy of the final.** You should work each problem here, enter the portions of work that are required in WeBWorK, scan or make an image of your work, and submit the work for each problem in Gradescope. **You do not need to do your work on a printout of this exam; it may be done on any plain paper. Your submitted work is essential for credit. If you have trouble submitting your work within 20 minutes of completing the test, please e-mail it to your lecture instructor and then continue to work to submit it in Gradescope.**

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1. [0 points] This problem is to ensure that you understand and agree to the conditions under which the exam is to be administered. In particular, you understand that:

\textit{This exam is to be completed without use of any physical aids (your textbook, notes, notecards, computers, calculators, etc.), external resources (the internet, etc.), or help from other individuals (other students, tutors, on-line help forums, etc.).}

For this problem, your upload should be a brief statement indicating that you agree to and are following the exam instructions, your signature, and an image of your UM ID.

2. [12 points] Suppose that we are solving a first-order, linear, homogeneous differential equation.

a. [6 points] If the general solution to the problem is \( y = Ce^{-\sin(2t)} \), what is the equation?

\textbf{Solution:} If the problem is first-order, linear, and homogeneous, then it is of the form \( y' + p(t)y = 0 \), and may be solved using an integrating factor \( \mu = \exp(\int p(t) \, dt) \): \((\mu y)' = 0\), so \( y = C\mu^{-1} \). We therefore know that \( \int p(t) \, dt = -\sin(2t) \), or \( p(t) = -2\cos(2t) \). Thus our equation is \( y' - 2\cos(2t)y = 0 \), or \( y = 2\cos(2t)y \).

b. [6 points] Does your equation have any critical points? Discuss why or why not, and the extent to which it makes sense (or doesn’t) to talk about the stability of the critical point.

\textbf{Solution:} Note that \( y = 0 \) is a solution, so this is a critical point—for a linear homogeneous problem this will always be a critical point. However, in general, because the equation is not autonomous, it does not make sense to think about stability: consider a point above \( y = 0 \), say \( y = 1 \). Then \( y = e^{-\sin(2t)} \), and \( y' = -2\cos(2t)e^{-\sin(2t)} \). For \( 0 \leq t < \pi/2 \) this is negative, suggesting that solutions will tend toward \( y = 0 \); however, for \( \pi/2 < t < 3\pi/2 \), the derivative becomes positive and solutions move away. Thus our traditional thinking about stability doesn’t work in this case.

For this problem, you will enter the value of the critical point(s) in WeBWorK in addition to uploading your work and explanation.
3. [10 points] Write a system of two constant-coefficient, homogeneous, linear differential equations which has a single critical point \((0,0)\) and for which solutions include (only) the functions \(e^{-t}\) and \(e^{-3t}\). Explain how you get your solutions and how you know it is correct. Sketch a phase portrait for your system.

For this problem, you will enter the equations for the system in WeBWorK.

**Solution:** A system of two constant-coefficient, homogeneous, linear differential equations is of the form \(x' = Ax\). We know for the solutions to be proportional to \(e^{-t}\) and \(e^{-3t}\) we must have \(\det(A - \lambda I) = (\lambda + 1)(\lambda + 3) = \lambda^2 + 4\lambda + 3\). The simplest way for this to be the case is to take \(A = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}\); a more interesting solution would be to look for \((a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 + 4\lambda + 3\). With \(a = d = -2\), this requires \(bc = 1\), so we can take \(b = c = 1\), so that \(A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\). In the first case, eigenvectors are \(v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) (for \(\lambda = -1\)) and \(v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) (for \(\lambda = -3\)). Trajectories in the phase plane collapse fastest in the second direction, so that we get the phase portrait below.

For the more interesting case, the eigenvectors are just rotated 45° counter-clockwise.
4. [12 points] Consider the differential equation $y'' + 2y' + y = k$, where $k$ is a nonzero real constant.

a. [6 points] Which of the following could be the phase portrait for this equation? Explain.

Solution: Note that the characteristic polynomial for the complementary homogeneous equation is $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$, so there is only one root, $\lambda = -1$ to that and therefore only one straight-line trajectory in the phase plane. This means that the phase portrait is either (II) or (III). Then note that as $t \to \infty$, $y \to k$ and $y' \to 0$, so the correct phase portrait must be (II).

b. [6 points] If this is a model for a mass-spring system and we increase the friction in the system, how would the phase portrait of the system change? Sketch a representative phase portrait in this case.

Solution: If the damping increases, the coefficient on $y'$ will increase, which will have the effect of changing the characteristic polynomial from $p(\lambda) = (\lambda + 1)^2$ to $p(\lambda) = \lambda^2 + (2 + a)\lambda + 1$, for some $a > 0$. Thus we have $\lambda = -\frac{1}{2}(2 + a) \pm \frac{1}{2}\sqrt{4a + a^2}$, which gives two real negative roots. Thus the phase portrait will have two straight-line trajectories, and will look like (IV) in the list above.

For this problem, you will enter the correct phase portrait for part (a) in WeBWorK.
5. [12 points] Suppose that the solution to a nonhomogeneous, linear, constant coefficient, second order differential equation with discontinuous forcing is given by the graph below. For $0 \leq t < 3\pi/2$, the solution is $y = 2e^{-t}\sin(2t)$. For $t \geq 3\pi/2$, the solution is $y = 0$.

![Graph of the solution](image)

**a.** [4 points] If the equation and initial conditions are $y'' + 2y' + ky = f(t)$, $y(0) = y_0$, $y'(0) = v_0$, what are the constants $k$, $y_0$, and $v_0$?

**Solution:** Note that from the form of the solution starting at $t = 0$, we know that the characteristic polynomial has roots $\lambda = -1 \pm 2i$, so that $p(\lambda) = (\lambda + 1)^2 + 4 = \lambda^2 + 2\lambda + 5$. Thus $k = 5$. From the graph or the function, $y(0) = y_0 = 0$. Then, from the function, we have $y' = -2e^{-t}\sin(2t) + 4e^{-t}\cos(2t)$, so that $y'(0) = v_0 = 4$.

**b.** [8 points] Propose a forcing $f(t) = a\delta(t - t_0)$ that will result in the indicated solution. Explain how you know your answer is correct, and why it will result in the solution given.

**Solution:** We know that at $t = 3\pi/2$, the solution becomes zero. We can accomplish this by letting $f(t) = c\delta(t - 3\pi/2)$ and picking $c$ so that at that point the slope of the function is zero. Because $y(3\pi/2) = 0$, this will result in the solution satisfying the differential equation with initial conditions $y(3\pi/2) = 0$ and $y'(3\pi/2) = 0$, so that the solution will be identically zero.

We have $y'(t)$ above. At $t = 3\pi/2$, $y'(3\pi/2) = -4e^{-3\pi/2}$, so we need $a = 4e^{-3\pi/2}$. This will discontinuously change $y'$ to zero, obtaining the result we desire.
6. [12 points] Consider the system of differential equations given by $x' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & -1 & 2 \end{pmatrix} x$.

   a. [5 points] Determine whether the vector functions 
   
   $x_1 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} e^{-t}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^t, x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t$
   
   form a fundamental solution set for this system.

   **Solution:** To determine if these are a fundamental solution set, we need either to verify that they are solutions and that they are linearly independent; or find the eigenvalues and eigenvectors of the system to confirm that these are linearly independent solutions. We do both here, but only one is necessary.

   First: are these solutions? Taking the coefficient matrix times the first function, $x_1$, we have 
   
   $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} e^{-t} = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix} e^{-t} = -x_1 = x_1'$. Thus this is a solution.

   Similarly, for $x_2$, we have $Ax_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^t = x_2'$, and for $x_3$, $Ax_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t = x_3'$, so they are all solutions. We can then check independence with the Wronskian:

   
   
   $W[x_1, x_2 x_3] = | \begin{pmatrix} 0 & 0 & e^t \\ 3e^{-t} & e^t & 0 \\ e^{-t} & e^t & 0 \end{pmatrix}| = (3 - 1)e^t = 2e^t \neq 0$.

   Thus they are linearly independent, and we have a fundamental solution set.

   Alternately, we see that the characteristic polynomial is $p(\lambda) = \det(A - \lambda I) = (1 - \lambda)((-2 - \lambda)(2 - \lambda) + 3) = (1 - \lambda)(\lambda^2 - 1) = (1 - \lambda)(\lambda - 1)(\lambda + 1)$, so that $\lambda = 1$ (twice) or $\lambda = -1$. If $\lambda = 1$, the eigenvectors satisfy $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & -1 & 1 \end{pmatrix} v = 0$, and we see that there are two solutions, corresponding to $x_2$ and $x_3$, above. Finally, if $\lambda = -1$, the eigenvector satisfies $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \end{pmatrix} v = 0$, so that $v = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, corresponding to $x_1$ above. Because we obtained these from the eigenvalues and eigenvectors we know that the solutions are linearly independent in this case, and therefore must be a fundamental solution set.

   b. [2 points] Write a general solution to the system.

   **Solution:** This is just $x = c_1 x_1 + c_2 x_2 + c_3 x_3$.

   c. [5 points] For what initial conditions will solutions to the system always remain bounded? Explain what this looks like and how you know it is correct.
Solution: The solution will only be bounded if it does not contain one of the terms with the growing exponential. Therefore we must have \( x(0) = c_1 \begin{pmatrix} 0 & 3 & 1 \end{pmatrix}^T \), for any value of \( c_1 \). Thus bounded solutions are restricted to the line through the origin in the direction of this eigenvector.
7. [12 points] Consider the predator-prey model

\[ x' = 4x - x^2 - \frac{1}{3}xy, \quad y' = -y + \frac{1}{3}xy. \]

By doing an analysis of this system by linearization and drawing a phase portrait, determine what you expect the long-term values of the populations will be. It may be useful to note that \( A = \begin{pmatrix} -3 & -1 \\ 1 & 0 \end{pmatrix} \) has eigenvalues \( \lambda \approx -2.6, -0.4 \), with corresponding eigenvectors \( v = (-2.6, 1)^T \) and \( v = (-0.4, 1)^T \).

**Solution:** We expect that the system will end up at any stable equilibrium, and to determine these must find the critical points and their linear stability. We note that the system has polynomial nonlinearities, thus is infinitely differentiable, and so is almost linear. Critical points are constant solutions, so, from the second equation, \( y' = 0 = -y(1 - \frac{1}{3}x) \), requiring \( y = 0 \) or \( x = 3 \). If \( y = 0 \), the first equation requires that \( x = 0 \) or \( x = 4 \); and if \( x = 3 \), the first equation requires \( y = 3 \). Therefore critical points are \((0,0), (4,0), \) and \((3,3)\).

To linearize, we consider the Jacobian, \( J = \begin{pmatrix} 4 - 2x - y/3 & -x/3 \\ y/3 & -1 + x/3 \end{pmatrix} \). At \((0,0)\), this is \( J(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \), which has eigenvalues \( \lambda = 4 \) and \( \lambda = -1 \). This is therefore an unstable saddle. Eigenvectors are along the axes, with outward trajectories along \( x \) and inward trajectories along \( y \). At \((4,0)\), \( J(4,0) = \begin{pmatrix} -4 & -4/3 \\ 0 & 1/3 \end{pmatrix} \), with eigenvalues \( \lambda = -4 \) and \( \lambda = \frac{1}{3} \) (because the matrix is triangular, we can read the eigenvalues from the diagonal). If \( \lambda = -4 \) the eigenvector is on the \( x \) axis, and trajectories converge to \((4,0)\), and if \( \lambda = \frac{1}{3} \), \( v = (4, -13)^T \). Thus this is also an unstable saddle.

Finally, at \((3,3)\), we have \( J(3,3) = \begin{pmatrix} -3 & -1 \\ 1 & 0 \end{pmatrix} \), so that we may use the useful observation to determine the eigenvalues and eigenvectors. Thus this has two negative eigenvalues, and is stable. We therefore expect that \((x,y) \to (3,3)\) as \( t \to \infty \). Putting this together with the preceding work, we have the phase portrait shown below.