Review of Complex Numbers

Introduction

This is a short review of the main concepts of *complex numbers*. Complex numbers are used throughout mathematics and its applications. In particular, when we try to solve differential equations it is often convenient and natural to use complex numbers to express the solutions. Here we review those ideas and results from the theory of complex numbers that will be used in Math 216.

A complex number z may be expressed as an ordered pair of *real* numbers:

$$z = (x, y) = x + iy$$

where $i := \sqrt{-1}$ (so $i^2 = -1$) and x and y are real numbers. The following notations are often used:

$x = \mathrm{Re}(z)$ or $x = \Re(z)$ denotes the *real part* of the complex number z
$y = \mathrm{Im}(z)$ or $y = \Im(z)$ denotes the *imaginary part* of the complex number z

Recall that two complex numbers are equal if and only if both the real and the imaginary parts are equal. In other words, $z_1 := (x_1, y_1)$ equals $z_2 := (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

A convenient way of thinking about complex numbers is to imagine them as points in the (x, y) plane (in this case it is called the *complex plane*), as illustrated in the following figure. In the complex plane, the line $y = 0$ is frequently called the *real axis*, and the line

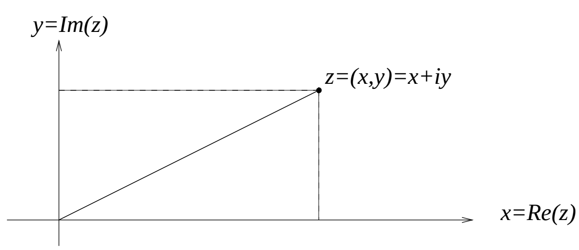

Figure 1: The complex number $z = x + iy$ plotted in the complex plane.

$x = 0$ is frequently called the *imaginary axis*.

Doing arithmetic with complex numbers

Addition and multiplication of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined by the following rules:

- Addition: $z_1 + z_2 := (x_1 + x_2, y_1 + y_2) = (x_1 + x_2) + i(y_1 + y_2)$.
- Multiplication: $z_1 z_2 := (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

Note that if we interpret z_1 and z_2 as points in the complex plane as in Figure 1, then addition of complex numbers is the same as vector addition in the plane; we are just adding the real and imaginary parts componentwise. On the other hand, the multiplication of two complex numbers may perhaps seem different than what you might have expected it to be; this is only an illusion, however, and when we introduce exponential forms for complex numbers later, the multiplication will make perfect sense.

Although complex numbers obey different rules of arithmetic than do ordinary real numbers, it is very important to keep in mind that the complex numbers simply generalize the notion of the real numbers. Indeed, we can think of the real number x as the complex number $(x, 0) = x + i0$. Such a complex number whose imaginary part is zero is said to be *purely real*. If we add or multiply two purely real complex numbers, then according to the rules for complex arithmetic, we have

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \quad \text{and} \quad (x_1, 0)(x_2, 0) = (x_1 x_2, 0)$$

so in each case the result is also a purely real complex number, and the real part in each case is exactly what we would have found by applying the usual rules of addition and multiplication for real numbers to the real parts. This shows that all the new operations defined for complex numbers when applied to purely real numbers give the usual familiar corresponding operations.

One way to think of $(0, 1)$ is as the *new* number i which is *purely imaginary* in the sense that its real part is zero, and so $(x, y) = x + iy$ is the sum of the purely real number x and the purely imaginary number iy.

Example: According to the above arithmetic rules for complex arithmetic, we have

$$(x, 0) + (0, y) = (x, y)\,, \quad \text{and} \quad (0, 1)(y, 0) = (0, y)\,.$$

Combining these, we deduce that

$$(x, y) = (x, 0) + (0, 1)(y, 0)$$

which is another way of writing the relation $z = x + iy$. □

Example: We can calculated repeated products of a complex number z with itself, which is what we mean by raising z to an integer power. Thus by definition really, $z^2 = zz$ and $z^3 = zzz$ and so on. In particular, $i^2 = ii = (0, 1)(0, 1)$. Using the rule for multiplication, we then see that

$$i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1$$

which verifies the fact that i is a square root of -1. □

Example: The fact that $i^2 = -1$ makes the rule for multiplication of complex numbers very easy to remember if one uses the $z = x + iy$ notation. Indeed just by multiplying out the individual terms,

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2$$

and then using $i^2 = -1$ we get

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

which is the rule for multiplication of complex numbers. □

It is easy to check directly from the definitions given of addition and multiplication of complex numbers that all of the familiar algebraic properties that we are familiar with hold for complex numbers too. In other words, complex arithmetic obeys the following rules:

- Commutative Law of Addition: $z_1 + z_2 = z_2 + z_1$
- Associative Law of Addition: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- Commutative Law of Multiplication: $z_1 z_2 = z_2 z_1$
- Distributive Law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
- Unique Additive Identity $0 = (0, 0) : z + 0 = 0 + z = z$
- Unique Multiplicative Identity $1 = (1, 0) : z \cdot 1 = 1 \cdot z = z$
- Additive inverse: $-z = (-x, -y) = -x - iy : z + (-z) = 0$
- Multiplicative inverse: For every complex number $z = (x, y) \neq 0$ there exists a complex number $w = (u, v)$ such that $(x, y)(u, v) = (u, v)(x, y) = (1, 0)$

It turns out that the multiplicative inverse of a nonzero complex number $z = (x, y)$ is the complex number

$$\left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2} \right)$$

which we denote by $1/z$. Now we can define the quotient of two complex numbers:

$$\frac{z_1}{z_2} := z_1 \cdot \frac{1}{z_2} = \frac{1}{z_2} \cdot z_1 \, .$$

Example: The multiplicative inverse of $i = (0, 1)$ is, according to the above formula,

$$\frac{1}{i} = (0, -1) = -i \, .$$

Therefore,

$$\frac{2}{i} = 2 \cdot \frac{1}{i} = -2i \, .$$

As a more complicated example, since

$$\frac{1}{1 + i} = \left(\frac{1}{2}, -\frac{1}{2} \right) = \frac{1}{2} - \frac{1}{2} i \, ,$$

we have

$$\frac{2-3i}{1+i} = (2-3i)\cdot\left(\frac{1}{2}-\frac{1}{2}i\right) = 1 - i - \frac{3}{2}i + \frac{3}{2}i^2 = -\frac{1}{2}-\frac{5}{2}i$$

because $i^2 = -1$. □

Why bother with complex numbers at all? Complex numbers were originally invented as an extension of real numbers in order to have a number system in which all polynomials have roots. For example, the equation $x^2 - 3x + 2 = 0$ has two real solutions, $x = 1$ or $x = 2$. But the similar-looking quadratic equation $x^2 - 3x + 3 = 0$ does not have any real roots at all! However, if we are willing to accept complex numbers as roots, then this quadratic equation also has two roots, namely the complex roots $3/2+i\sqrt{3}/2$ and $3/2-i\sqrt{3}/2$. More generally, for $a_n \neq 0$, and other given numbers $a_0, \ldots, a_{n-1}$, the polynomial equation $a_n x^n + ... + a_1 x_1 + a_0 = 0$ of degree n does not necessarily have any real solutions. However, complex numbers enjoy the property that if $a_0, a_1, ..., a_n$ are complex numbers and $a_n \neq 0$, then $a_n z^n + ... + a_1 z_1 + a_0 = 0$ always has n solutions (although not all roots are necessarily distinct). This fact is known as the *Fundamental Theorem of Algebra*. In other words, to find the solutions of a polynomial equation, you never need to look further than the complex numbers (remarkably, this is true even if the coefficients a_k are themselves generalized from real numbers to complex numbers). *This is exactly why we need complex numbers in a course on differential equations like Math 216: they are necessary to give us all of the roots of the characteristic polynomial that arises from seeking exponential solutions proportional to* e^{rt} *of a constant-coefficient differential equation, or system of differential equations.*

Some additional terminology for complex numbers

Associated with each complex number z is a positive number called the *absolute value* or *modulus* of z and written as $|z|$; the definition in terms of the real and imaginary parts of $z = x + iy$ is

$$|z| := \sqrt{x^2 + y^2}\,.$$

If we visualize z as a point in the complex plane as in Figure 1, the modulus of z is just the distance from the point (x, y) to the origin $(0, 0)$. See Figure 2. Unless z_1 and z_2

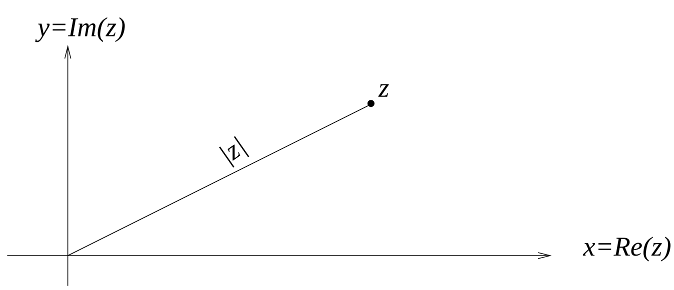

Figure 2: The modulus of a complex number $z = x + iy$.

are purely real, an inequality like "$z_1 < z_2$" has no meaning because somehow both real and imaginary parts would have to be compared. But the inequality $|z_1| < |z_2|$ does have meaning; according to Figure 2 it means that z_1 is closer to $(0, 0)$ than z_2 is.

Example: According to the definition, $|(1, 2)| = \sqrt{1^2 + 2^2} = \sqrt{5}$. □

Next, for each complex number z, there is another complex number called the *complex conjugate* of z and denoted by $\overline{z}$ or z^*. The complex conjugate of $z = (x, y) = x + iy$ is defined by

$$\overline{z} = z^* := (x, -y) = x - iy\,,$$

so taking the complex conjugate of a complex number z amounts to changing the sign of its imaginary part. Geometrically, this amounts to reflection of the point representing z in the complex plane through the real axis, as shown in Figure 3.

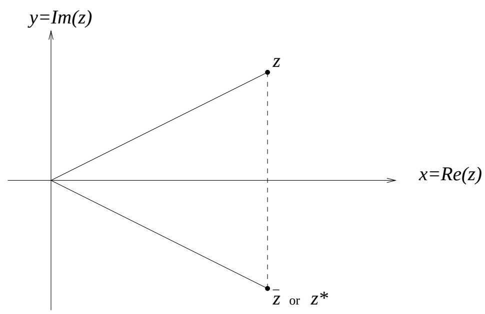

Figure 3: The complex conjugate of a complex number z.

Example: According to the definition, $\overline{1} = \overline{(1, 0)} = (1, 0) = 1$. Similarly, $i^* = (0, 1)^* = (0, -1) = -i$. □

Example: The following identities are easy to establish using the definition:

$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}\,,$$

$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}\,,$$

and

$$\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}\,.$$

Also, $\overline{\overline{z}} = z$, in other words, the complex conjugate of the complex conjugate of any number z is z itself. Why are there two different notations for the complex conjugate of a complex number? Generally, the bar notation is easier to read, unless it gets in the way of a dot on the i, or unless it is easily confused with the line separating the numerator and denominator of a fraction; for these situations, we have the option of using the star superscript. □

Example: Note that if $z = (x, y) = x + iy$, then

$$z + \overline{z} = (x, y) + (x, -y) = (2x, 0) = 2 \cdot \mathrm{Re}(z)\,,$$

$$z - \overline{z} = (x, y) - (x, -y) = (0, 2y) = 2i \cdot \mathrm{Im}(z)\,.$$

That is, we have that

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}) \quad \text{and} \quad \mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})\,.$$

This gives us a simple way to express the real and imaginary parts of z in terms of z and its complex conjugate. □

Example: Again, directly from the definition of complex conjugation,

$$zz^* = (x + iy)(x - iy) = x^2 - ixy + ixy + y^2 = x^2 + y^2 = |z|^2\,,$$

where in the last step we used the definition of the modulus of z. It follows that by multiplying the numerator and denominator of $1/z$ by z^*, we get

$$z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2}\,.$$

This is one way to deduce the formula we gave earlier for $1/z$. □

Polar form for complex numbers

Points in the complex plane can be identified by their Cartesian coordinates (x, y), or by their polar coordinates (r, θ) as indicated in Figure 4. Elementary trigonometry tells us that

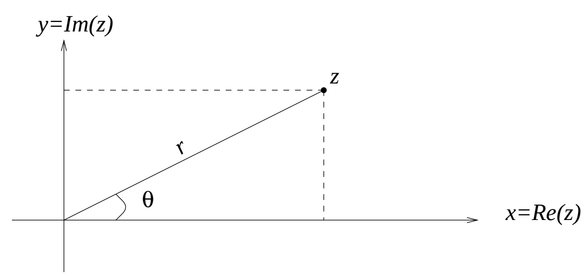

Figure 4: The polar coordinates of the complex number z.

the Cartesian and polar coordinates are related by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. We may therefore write the complex number z in terms of its polar coordinates in the following way:

$$z = x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta))\,.$$

Geometrically, it is easy to see that the modulus $|z|$ of z is the same thing as the polar coordinate r; however it is also easy to see this from the above formula using the trigonometry identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$:

$$|z| = \sqrt{r^2\cos(\theta)^2 + r^2\sin(\theta)^2} = r\sqrt{\cos^2\theta + \sin^2\theta} = r\,.$$

Example: For $z = 1 - i$ we have $r = \sqrt{2}$ and $\theta = -\pi/4$, and therefore

$$1 - i = \sqrt{2}[\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})] .$$

The angle θ is not unique, but its possible values differ from each other by multiples of 2π. For example, $\theta = 2\pi n - \pi/4$ works too, for any $n = 0, \pm 1, \pm 2, \ldots$. □

The angle θ is called the *argument* or *phase angle* of z and is denoted

$$\theta = \arg(z) .$$

Again thinking geometrically, it is easy to see that for a complex number to have an argument, it must be nonzero, which is the same thing as saying that its modulus is nonzero, or that $r \neq 0$. Since for $z \neq 0$ there are many values of $\arg(z)$, it is useful to define a particular value, called the *principal value* of $\arg(z)$ and denoted by $\operatorname{Arg}(z)$ as the unique value of θ between $-\pi$ and π:

$$-\pi < \operatorname{Arg}(z) \leq \pi .$$

If we are given the polar coordinates (r, θ) of a complex number z, then it is straightforward to calculate the corresponding Cartesian coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$. Finding the polar coordinates given the Cartesian coordinates is a little more tricky. Of course it is easy to find $r = \sqrt{x^2 + y^2}$, but then we need to find an angle θ so that

$$\cos(\theta) = \frac{x}{r} , \quad \text{and} \quad \sin(\theta) = \frac{y}{r} .$$

It also follows from these relations that the angle θ we seek satisfies

$$\tan(\theta) = \frac{y}{x} = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} .$$

The way to find θ is to use the inverse trigonometric functions; however in doing so you need to make sure that the resulting angle is in the correct quadrant given the signs of x and y.

Example: Suppose that $z = x + iy = -1 + i$. To find the polar coordinates of z, we first calculate $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. Then, the angle θ we seek satisfies $\cos(\theta) = -1/\sqrt{2}$ and $\sin(\theta) = 1/\sqrt{2}$, or if we combine these, $\tan(\theta) = -1$. If we apply the arctangent function to solve for θ, we get $-\pi/4$, which is indeed an angle whose tangent is -1. However it is not an angle whose sine is $1/\sqrt{2}$. To get the right answer we recall that the arctangent function is only defined up to integer multiples of π, and therefore $\theta = 3\pi/4$ is also an angle whose tangent is -1. In this case, we also see that indeed $\cos(3\pi/4) = -1/\sqrt{2}$ and that $\sin(3\pi/4) = 1/\sqrt{2}$, so that indeed $\theta = \arg(z) = 3\pi/4$. Moreover, since $-\pi < 3\pi/4 \leq \pi$, we also have $\operatorname{Arg}(z) = 3\pi/4$. □

One of the reasons for introducing polar coordinates for complex numbers is that it gives a simple geometrical interpretation to the process of complex multiplication. Indeed, if

$$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)) , \quad \text{and} \quad z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2)) ,$$

then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \,(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2 \left[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i\,(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))\right] \\ &= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right] . \end{aligned}$$

We used two trigonometric identities in the last step above to arrive at a formula in terms of the sum of the angles $\theta_1 + \theta_2$. In particular, this calculation shows that

$$|z_1 z_2| = |z_1| \cdot |z_2| , \quad \text{and} \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) .$$

That is, when we multiply two complex numbers we multiply their moduli and add their arguments. Note also that if $r \neq 0$, then

$$\frac{1}{z} = \frac{1}{r(\cos(\theta) + i\sin(\theta))} \cdot \frac{r(\cos(\theta) - i\sin(\theta))}{r(\cos(\theta) - i\sin(\theta))} = \frac{1}{r}(\cos(\theta) - i\sin(\theta)) = \frac{1}{r}\left[\cos(-\theta) + i\sin(-\theta)\right] ,$$

so $|z^{-1}| = 1/|z|$ and $\arg(z^{-1}) = -\arg(z)$. Note also that $\arg(\overline{z}) = -\arg(z)$.

Exponential form of a complex number: Euler's formula

Recall that the infinite power series expansion defining e^w is

$$e^w = 1 + w + \frac{w^2}{2} + \frac{w^3}{3!} + \cdots .$$

It turns out that both sides make sense when w is a complex number, and in particular if $w = i\theta$ where θ is a real angle. In this case the series expansion becomes

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \cdots \\ &= \left[1 - \frac{\theta^2}{2} + \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \cdots\right] \\ &= \cos(\theta) + i\sin(\theta) , \end{aligned}$$

where at the end we grouped the purely real and purely imaginary terms and recalled the infinite power series expansions of $\cos(\theta)$ and $\sin(\theta)$ for real values of θ. The remarkable formula we have found in this way:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is known as *Euler's formula*. If we use Euler's formula, we can express z in terms of its polar coordinates in an even simpler form:

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta} ,$$

where $r = |z|$ is the absolute value of z and $\theta = \arg(z)$ is the argument. This is called the *exponential form* of the complex number z.

Now we can see that we can also view the effect of multiplying two complex numbers that are expressed in terms of their polar coordinates (multiplication of the moduli and addition of the arguments) as being a simple consequence of the rules for multiplying exponentials:

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}\,.$$

Also,

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \left(\frac{1}{r}\right) e^{-i\theta}\,,$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}\,.$$

Note that for any $n = 0, \pm 1, \pm 2, ...$ we have $z = re^{i(\theta+2\pi n)}$; also $\overline{z} = re^{-i\theta}$.

The exponential form of a complex number z makes it easy to compute powers of z. Indeed, $z^n = (re^{i\theta})^n = r^n e^{in\theta}$. Combining this with Euler's formula we have *DeMoivre's Theorem*, which gives the real and imaginary parts of any power of a complex number of modulus one:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)\,.$$

The proof of DeMoivre's theorem is basically one line long:

$$(\cos(\theta) + i\sin(\theta))^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)\,.$$

Euler's formula was used in the first and last steps.

Example: Writing out DeMoivre's formula for $n = 2$, and doing the multiplication, gives

$$\cos(2\theta) + i\sin(2\theta) = (\cos(\theta) + i\sin(\theta))^2 = \cos(\theta)^2 - \sin(\theta)^2 + i2\sin(\theta)\cos(\theta)\,.$$

Since the real and imaginary parts on both sides must be equal, we get $\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. This example shows that DeMoivre's Theorem provides an easy way to remember the multiple-angle trigonometry identities. □

Finding roots of complex numbers is also made easy using the exponential form. For example,

$$z^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{i\theta/2}\,,$$

but, since θ may be replaced by $\theta + 2\pi k$ for any integer k without changing z, we have more generally that

$$(re^{i(\theta+2\pi k)})^{1/2} = r^{1/2} e^{i(\theta/2+\pi k)}\,.$$

The right-hand side gives only two possible answers as k ranges over all possible integers. These two complex numbers are the two square roots of z. The nth roots of a complex number z are calculated in exactly the same way:

$$\begin{aligned} z^{1/n} &= r^{1/n} e^{i(\theta+2\pi k)/n} \\ &= r^{1/n} e^{i(\theta/n+2\pi k/n)}, \end{aligned}$$

and now we see that the right-hand side gives n distinct values as k ranges over the integers. Therefore every nonzero complex number z has n distinct nth roots.

Example: The distinct cube roots of 1 are $1^{1/3} = 1$ or $1^{1/3} = e^{2\pi i/3}$ or $1^{1/3} = e^{4\pi i/3}$. They are illustrated in the complex plane in Figure 5. □

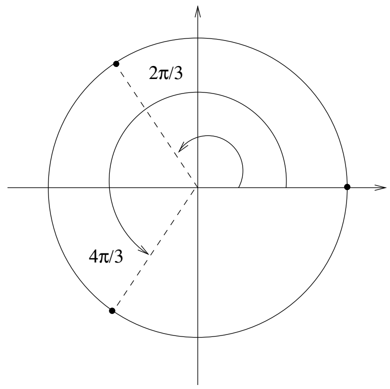

Figure 5: The cube roots of $z = 1$ in the complex plane.

Problems

1. Compute the following (express in the form $x + iy$):
   (a) $(2 + 3i)(5 + 7i)$
   (b) $(2 + 3i)^2$
   (c) $(2 + 3i) + (-5 - 7i)$
   (d) $(2i)^3 - (2i)^2 + 2i - 1$
   (e) $(5e^{i7\pi/6})^3$
   (f) $\sqrt{e^{i\pi/3}}$ (Give two answers)
2. Solve the following for z, expressed as $x + iy$:
   (a) $z^2 - 2z + 2 = 0$
   (b) $z^2 + z + 1 = 0$
3. Compute the following if $z = x + iy$ (express in terms of x and/or y):
   (a) $\mathrm{Im}(iz)$
   (b) $\mathrm{Re}(z)$
   (c) $\mathrm{Re}(iz)$
   (d) $-\mathrm{Im}(z)$

4. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially (sketch the plot):

   (a) $z_1 = 2i$ and $z_2 = 2/3 - i$
   (b) $z_1 = -\sqrt{3} + i$ and $z_2 = \sqrt{3}$
   (c) $z_1 = -3 + i$ and $z_2 = 1 + 4i$

5. Simplify the following:

   (a) $\overline{\overline{z} + 3i} =$
   (b) $\overline{i\overline{z}} =$
   (c) $\overline{(2+i)^2} =$

6. Sketch the set of points in the complex plane determined by the following:

   (a) $|z - (1 - i)| = 1$
   (b) $\text{Re}(\overline{z} - i) = 2$

7. Solve the following simultaneous equations for z_1 and z_2:

   (a) $z_1 + z_2 = 2$
   $(i - 1)z_1 + (1 + i)z_2 = 3$
   (b) $z_1 + z_2 = 2$
   $(i - 1)z_1 + (1 + i)z_2 = 0$

8. Determine $\text{Arg}(z)$ for each of the following:

   (a) $z = -2/(1 + \sqrt{3}i)$
   (b) $z = i/(-2 - 2i)$
   (c) $z = (\sqrt{3} - i)^6$

9. Use the exponential form to compute the following:

   (a) $i(1 - \sqrt{3}i)(\sqrt{3} + i) =$
   (b) $5i/(1 + i) =$
   (c) $(-1 + i)^7 =$

10. Use DeMoivre's Theorem to prove the following trigonometric identities:

    (a) $\cos(3\theta) = \cos(\theta)^3 - 3\cos(\theta)\sin(\theta)^2$
    (b) $\sin(3\theta) = 3\cos(\theta)^2\sin(\theta) - \sin(\theta)^3$

    Hint: Expand $(\cos\theta + i\sin\theta)^3$ and use DeMoivre's Theorem.

11. Find all the roots for the following and exhibit them geometrically:

    (a) $(2i)^{1/2} =$

(b) $(1-\sqrt{3}i)^{1/2} =$

(c) $(-1)^{1/3} =$

(d) $(-16)^{1/4} =$

Hint: Use $z^{1/n} = r^{1/n}e^{i(\theta+2\pi k)/n}$ for all $k = 0, \pm 1, \pm 2, \ldots$

12. If the roots of the equation $x^2 + 2bx + c = 0$ are the complex numbers $p \pm iq$ (p and q are real), find expressions for b and c in terms of p and q.

13. Check directly that the multiplicative inverse of a nonzero complex number $z = (x, y)$ is indeed given by

$$\frac{1}{z} = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2} \right) .$$

Hint: multiply this expression by $z = (x, y)$.

14. Show that for a nonzero complex number $z = (x, y)$, $|z^{-1}| = |z|^{-1} = 1/\sqrt{x^2+y^2}$.

15. Show that the equation $|z| = 1$ describes the set of points on the unit circle in the complex plane.

16. Show that $|i^2| = 1$ and $\arg(i^2) = \pi$.

17. Prove the identities given in the third example on page 5, by appealing directly to the definitions of complex conjugation and complex arithmetic.