1. Model and Objectives

1.1. Model. In this lab, we will study the Gompertz equation, a first-order ordinary differential equation which models the growth of cancerous tumors:

\[
\frac{dy}{dt} = ry \ln(K/y).
\]

The constants \( r \) and \( K \) in this equation are positive. The function \( y(t) \) gives the volume of the tumor at time \( t \).

1.2. Objectives. In this lab our goals are to see some connections with the material we’ve seen in calculus, understand how these may give us approximations to the solutions to differential equations, and explore the behavior of these solutions. In particular, we want to:

- Learn how we can approximate a nonlinear ordinary differential equation (ODE) with a simpler (usually linear) ODE by using a Taylor polynomial (the truncation of a Taylor series) to approximate the nonlinear terms.1
- See how Taylor series and Taylor polynomials can be used to approximate the solutions to an ODE.

Go back and reread those two objectives: notice how they are similar, and how they are different! Understanding this is key to completing the rest of the lab.

2. Pre-lab

2.1. Taylor series. In our calculus classes, we learned how to construct Taylor series: that is, for a function \( f(x) \) which has derivatives of all orders at a point \( x_0 \), we found a series of the form

\[
a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n = f(x).
\]

The series on the left-hand side we call the Taylor expansion of \( f(x) \) near \( x_0 \). Note that (2) is an equality, which means that \( f(x) \) and the series are the same function in some neighborhood of \( x_0 \).

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1This is an introduction to a fundamental idea we will be seeing in class throughout the rest of the semester: that we can gain a qualitative understanding of nonlinear equations locally, by linearization.
Example 1: The function $\sin x$ is equal to the following infinite series:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1},$$

and the equality holds for all real numbers $x$. What is $x_0$ in this case? What are the coefficients $a_n$? In particular, what is $a_2$?

The expansion is in terms of $(x - x_0)^k$, so $x_0 = 0$. The coefficients $a_n$ are zero for all even $n$ (and $n = 0$), and are $\pm1/n!$ for odd coefficients. The first, $a_1$, is positive, and subsequent terms alternate sign.

We can see how the series generates the function as we add terms to the sum; this is shown in the figure, below.

![Graph of Taylor polynomials](image)

We see that the Taylor polynomials obtained by truncating the series to different values of $n$ resemble the sine function better and better; if we continued to an infinite number of terms, we would have exact equality.

A consequence of the equality (2) that will be significant in this lab is that all derivatives of $f(x)$ and of the series are also equal as functions—in particular, they must be equal at the point $x = x_0$. This fact can be used to calculate the coefficients $a_n$, as we show in the following example.

Example 2: How are the values of $a_n$ related to $f$ and its derivatives?

Note that if we plug $x = x_0$ into both sides of (2) we are left with $f(x_0) = a_0$ (this tells us the value of $a_0$!). Taking the derivative of each side of (2), we have

$$f'(x) = 0 + a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

so that $f'(x_0) = a_1$, that is, $a_1 = f'(x_0)$. Continuing with the second derivative, we have

$$f''(x) = 2a_2 + 6a_3(x - x_0) + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

and $f''(x_0) = 2a_2$—so $a_2 = \frac{1}{2} f''(x_0)$. Another derivative gives us $a_3 = \frac{1}{6} f'''(x_0) = \frac{1}{3!} f'''(x_0)$. What happens as we continue taking derivatives?
Each derivative pulls down another factor from the exponent, so that after \( k \) derivatives we have \( a_k = \frac{1}{k!} f^{(k)}(x_0) \).

Thus, Example 2 tells us the coefficients of a Taylor series of a known function \( f \), and thus that the series (2) is

\[
f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n.
\]

In practice, of course, it’s difficult to work with a series which has an infinite number of terms, so we will often truncate the series to some number of terms and use that as an approximation to the function we want. In particular, as we will see repeatedly in this course, linear approximations often allow us to work with problems that would otherwise be intractable.

**Exercise 1:** The Taylor series for \( \ln(y) \) about \( y = 1 \) is

\[
\ln(y) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} (y - 1)^n
\]

for \( y - 1 \in (-1, 1] \) (that is, \( y \in (0, 2] \)). What polynomials do we get if we truncate this series at \( n = 1 \) \( n = 2 \) \( n = 0 \) (hint: the \( n = 0 \)th approximation is defined!)? Compare the value of each of these with that of \( \ln(y) \) at \( y = 1.1 \) and \( y = 1.75 \). Note how the error differs at the different \( y \) values.

We call the approximations we found in Exercise 1 **Taylor polynomials**, for obvious reasons. When we truncate the Taylor series to a polynomial of degree \( n \), we say the approximation is “of order \( n \)” or “\( n \)th order.”

2.2. **Approximating nonlinear differential equations.** We can use Taylor polynomials to find approximations to differential equations, as shown in Exercise 2:

**Exercise 2:** Use the Taylor polynomials from Exercise 1 to find approximations to the right hand side of the Gompertz equation of orders \( n = 0 \), \( n = 1 \), and \( n = 2 \). (Note that \( \ln(K/y) = \ln(K) - \ln(y) \).)

In general, we will use this technique to obtain a **linear equation**, which we can then solve. The solution that we obtain to this simpler equation will be a good approximation to the solution to the original equation near the expansion point \( x_0 \) (here, \( y_0 = 1 \)), as suggested by our work in Exercise 1.

2.3. **Series solutions to linear differential equations.** Another way we can use Taylor series is to look for a series that is the solution to a differential equation. For example, if we have \( y' = cy \), with \( y(0) = 1 \), then we are saying there is a function \( y(t) \) that satisfies this equation and initial condition. If this function has a convergent Taylor series \( y(t) = \sum_{n=0}^{\infty} a_n t^n \) in a neighborhood of
the initial condition, we should be able to follow the steps of Example 2 to find the \( a_n \) and thus find the solution as a Taylor series. Let’s do that here.²

**Example 3: Find the solution to** \( y' = cy \), \( y(0) = 1 \), **as a Taylor series.**

Let \( y(t) = \sum_{n=0}^{\infty} a_n t^n \). From Example 2, we know that \( a_0 = y(0) \), \( a_1 = y'(0) \), and, in general, \( a_n = \frac{1}{n!} y^{(n)}(0) \). The initial condition therefore tells us that \( a_0 = 1 \). To find later coefficients, we can plug the series into the equation and then match powers of \( t \). Differentiating the series to get the left-hand side of the equation, \( y' = \sum_{n=1}^{\infty} n a_n t^{n-1} \), so we have

\[
a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots = c(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots).
\]

Matching powers of \( t \), we have \( a_1 = c a_0 = c \), \( a_2 = \frac{1}{2} c a_1 = \frac{1}{2} c^2 \), \( a_3 = \frac{1}{3} c a_2 = \frac{1}{6} c^3 \), and so on: \( a_n = \frac{1}{n!} c^n \). Putting this together, we have our Taylor series solution,

\[
y(t) = \sum_{n=0}^{\infty} \frac{1}{n!} c^n t^n.
\]

**Exercise 3:** Use separation of variables to find the unique solution to the initial value problem \( y' = r \ln(K) y \) with \( y(0) = 1 \). The Taylor series (4) also gives a solution to this initial value problem (with \( c = r \ln(K) \)). How many solutions have we found?

### 2.4. Series solutions to nonlinear differential equations.

The logistic equation

\( y' = \alpha y - \beta y^2 \)

and Gompertz equation (1), \( y' = ry \ln(K/y) \), are examples of nonlinear autonomous differential equations. Our approximation of the Gompertz equation from Exercise 2 with \( n = 1 \) is a logistic equation. If we try to calculate a solution to this equation in the form of the series \( y(t) = \sum_{n=0}^{\infty} a_n t^n \), the resulting calculations for the coefficients \( a_n \) become more difficult to reduce to closed form. In particular, there is a \( y^2 \) term in the logistic equation, so we would have to calculate \((\sum_{n=0}^{\infty} a_n t^n)^2\) to find the coefficients \( a_n \).

For the Gompertz equation, plugging in \( y(t) = \sum_{n=0}^{\infty} a_n t^n \) is even worse! We end up with the term

\[
\ln \left( \frac{K}{y} \right) = \ln \left( \frac{K}{\sum_{n=0}^{\infty} a_n t^n} \right) = \ln(K) - \ln \left( \sum_{n=0}^{\infty} a_n t^n \right)
\]

on the right-hand side of the equation.

²Why would we do this? Well, in general we probably wouldn’t, because we know that all linear first-order differential equations can be solved with integrating factors. But we might try a series solution (or polynomial approximation) if the resulting integral is not something we can find in closed form. And for non-constant coefficient equations, which we can’t always solve, this approximation becomes more interesting.
In the following exercises we explore the issues associated with finding series solutions to these, though, you will be pleased to hear, we will not grind through the details of their solution. Instead, our goal is to understand how it can work, without getting too bogged down in the details.

**Exercise 4:** Consider a specific example of the logistic equation, $y' = 3y - y^2$, with $y(0) = 1$. Let $y = \sum_{n=0}^{\infty} a_n t^n$. Find $y'$ and plug it and $y$ into the equation, following Example 3, to find $a_0$, $a_1$, $a_2$, and $a_3$. Note that to deal with the quadratic term $y^2$ you will have to find the first few terms in the product $(\sum_{n=0}^{\infty} a_n t^n)^2 \approx (a_0 + a_1 t + a_2 t^2)^2$.

**References**