Lab 5: Nonlinear Systems

Goals

In this lab you will use the `pplane7` program to study two nonlinear systems by direct numerical simulation. The first model, from population biology, displays interesting nonlinear oscillations (so-called *limit cycles*). The second is a system whose solutions depend on a parameter. Neither of these systems is described by exactly solvable systems of differential equations. Although much may be learned from strictly theoretical analyses, we must ultimately rely on computational methods to extract quantitative predictions from these systems.

Application 1: predator-prey species interactions

In class we considered a model of predator-prey species interactions known as the Lotka-Volterra model (referred to in Section 6.3 as the *predator-prey system*). If x describes the size of a population of rabbits and y describes a population of foxes (which feed on the population of rabbits) then the Lotka-Volterra model of their interactions says that there are positive constants a, b, c, d so that

$$\frac{dx}{dt} = x(a - by)\,, \tag{1}$$

$$\frac{dy}{dt} = y(-c + dx)\,. \tag{2}$$

That is, the exponential growth rate of rabbits is decreased by the presence of foxes and the exponential death rate of foxes is decreased by the presence of rabbits. This model predicts some unlikely behavior. In the absence of foxes ($y = 0$), equation (1) becomes $dx/dt = ax$. In other words, without any foxes the rabbits will always grow exponentially without bound. And even if the predator population is small, they will always eat the prey at a rate proportional to their product. In other words, 10 foxes surrounded by 100,000 rabbits would each have to eat ten times more than 10 foxes surrounded by 10,000 rabbits. If the rabbit population could be held at a fixed level $x_0 > c/d$, equation (2) becomes $dy/dt = Cy$ where $C = -c + dx_0 > 0$. In other words, if the rabbit population is maintained at a given level, above some threshold, the fox population will always grow exponentially without bound. None of these predictions are ecologically reasonable. The following model addresses these problems.

For positive values of r, a more reasonable model of the two populations is the system

$$\frac{dx}{dt} = x(1 - x) - \frac{5xy}{5x + 1}\,, \tag{3}$$

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{x}\right). \tag{4}$$

In the absence of predators, the prey satisfies the logistic equation with equilibrium population $x = 1$. In the presence of predators, prey is consumed at a rate $5xy/(5x+1)$. That is, if x is large compared to $1/5$, then $5xy/(5x+1) \approx y$, and the predators consume prey at a rate proportional to the predator population. On the other hand, if x is small compared to $1/5$ then $5xy/(5x+1) \approx 5xy$ and only then do the predators consume prey at a rate proportional to xy as in the Lotka-Volterra model. Furthermore, if the prey population x is held fixed somehow, we ignore the differential equation governing $x(t)$ and replace $x(t)$ by a constant, and we then see that the predator population satisfies a logistic growth equation with equilibrium population x (the predators and prey are equal in number in equilibrium). For this logistic growth model, the parameter r is the inverse relaxation time for the predator population, that is, $1/r$ is proportional to the time it takes the predator population to equilibrate. (Note: We have already scaled the variables and chosen some parameter values in equations (3) and (4). The general version of the model would have many more parameters.)

Application 2: bifurcation

When one tries to understand the behavior of a nonlinear system one of the first things one looks at is the set of equilibrium solutions. The number and type of equilibrium solutions may well depend on some parameter(s) of the system: the mass of a component, the stiffness of a spring, the length of a lever, the resistance of an electronic component, and so on. In this section of the lab you will observe in a very simple case how the number of equilibrium points of a system of differential equations changes as a parameter varies. A change in the number of equilibrium points, or in the stability of the equilibria, usually leads to a dramatic qualitative change in the behavior of the non-equilibrium solutions as well. Such a qualitative change is called a *bifurcation* and the associated parameter value at which the number or stability of equilibrium points changes is called a *bifurcation point*. A model nonlinear system of differential equations that exhibits bifurcations as a parameter varies is

$$\frac{dx}{dt} = ax - y\,, \tag{5}$$

$$\frac{dy}{dt} = x + ay + x^2\,. \tag{6}$$

In this system a is the parameter.

Prelab assignment

Before arriving in the lab, answer the following questions. You will need your answers in lab to work the problems, and your recitation instructor may check that you have brought them. These problems are to be handed in as part of your lab report.

1. Find all of the fixed points (critical points, or equilibria) of the modified predator-prey system, equations (3) and (4). Calculate the numerical value of the coexistence point corresponding to positive values of x and y.

2. By setting $dx/dt = 0$, find the curve in the phase plane where the trajectories of (3) and (4) are vertical, and by setting $dy/dt = 0$, find the curve in the phase plane where the trajectories are horizontal. Note that these two curves, called respectively the *x-nullcline* and the *y-nullcline*, are *not* trajectories themselves, but since it is known that any trajectory that crosses the x-nullcline does so vertically, and that any trajectory that crosses the y-nullcline does so horizontally, sketching these non-solution curves can help in sketching the phase portrait. Use the information from problem 1 along with these curves to sketch possible phase portraits for both small and larger values of r (to be concrete, think about $r = 0.01$ as being small, and $r = 0.1$ as being larger). Your sketches should be qualitatively different for different values of r.

3. The system exhibits very different behavior depending on whether $r > r_c$ or $r < r_c$, where $r_c = .053576\ldots$. In one regime, the coexistence point is stable and all solutions are attracted to it. In the other region, the coexistence point is unstable and population levels starting near the point spiral outward. Which do you think happens for which values of r? That is, do you think that a large or a small value of r ought to correspond to the stable coexistence or to the oscillatory behavior? (Either provide a coherent logical argument based on your solutions to the previous problems, or do a stability analysis of the coexistence fixed point to justify your prediction.)

4. Find the equilibrium points for the second system; that is, for equations (5) and (6).

In the lab

We will be using Matlab with the program `pplane7` to study phase portraits for nonlinear systems. The program `pplane7` is an analogue of `dfield7` that we used in Lab 1 that is specially adapted to systems of two differential equations. Using `pplane7`, we can let Matlab handle the numerical approximation of solutions and instead focus on the meaning of the solutions and the way that solutions change as parameters are varied.

Studying phase portraits using `pplane7`

To study the evolution of the fox and rabbit populations over time, you will want to generate a phase portrait plotting x against y. The following describes how you are to use `pplane7` to generate these phase portraits.

After you log onto the Macintosh system in the Instructional Computing Lab in the basement of East Hall, or onto your own system, launch Matlab and type `pplane7` in the Command Window and hit return. As happened with the `dfield7` program you used in Lab 1 a window with the title PPLANE7 Setup will open with lots of little boxes all filled in; ignore them for now and click on the Proceed button. You will see a graph with a direction field corresponding to a *system* of differential equations. Put the cursor on any point and click. You have just chosen initial conditions for the system. Now you know what the solution to the system of differential equations with your choice of initial conditions looks like. Plot a few solutions in this way. If your graph is getting too cluttered go to the Edit menu on the PPLANE7 Display window and select Erase all solutions.

After you have plotted a few solutions in the PPLANE7 Display window, go to the Graph menu on that window and select y vs t. Your cursor will become cross hairs; center the cross hairs on a solution curve and click. (Be sure to center the cross hairs on a solution curve you have already plotted, otherwise no graph will appear and you will see an error message at the bottom of the PPLANE7 Display window telling you that This object is not a solution curve.) You will see the corresponding plot of y versus t in a new window. You can change your mind and click on x vs t, or both, or 3D, and so on, using the Graph menu to the right of the plot in the new window.

Finally, go to the Solutions menu on the PPLANE7 Display window and choose Find an equilibrium point; your cursor will again become cross hairs, and if you position the cross hairs near an equilibrium point and click you will get a red dot at the equilibrium point and some additional useful information concerning the nature of the equilibrium point in a small box. You can repeat the command and find another equilibrium point for this system.

When you use `pplane7` to do this lab you will, of course, need to change the system in the PPLANE7 Setup window to the system you want to study. In the first case, the modified predator-prey system, there is a parameter r in the system. You can enter the equations with the symbol r in them and then, below the equations box, enter $r = 0.5$ (or whatever value you wish) in the parameters or expressions box. (A parameter is a constant of the problem that may however change its value from one problem to the next.)

Some important points: when you set up your equation you also enter the minimum and maximum values for x and y as you did with `dfield7` but it is often more convenient to zoom in or zoom back. You will find those commands under the Edit menu. In addition you should have the solver evolve the solution *forward* in time. This can be done by changing the solution direction in the Options menu for the PPLANE7 Display window. (Looking at the solution only in the forward direction tells you whether solutions are moving towards or away from an equilibrium point.) Now you have the tools to do the lab.

Lab problems

1. Use `pplane7` to solve the system of equations (3) and (4) and print phase portaits. Start from various initial conditions, and use $r = 0.07$, $r = 0.05$, and $r = 0.03$. You should zoom in on important features you see in the phase portraits.

2. Check your prediction from Prelab problem 3. What, if anything, surprised you about the behavior of the system?

3. What is different about the oscillatory state here compared to that of the (simpler, but less realistic) Lotka-Volterra system of differential equations (1) and (2)? Discuss.

(Hint: Consider the dependence of the steady-state oscillation amplitude on the initial condition. How many different closed orbits do you see for each value of r?)

4. Classify the equilibrium points of the second system (equations (5) and (6)) when $a = -0.5$, $a = 0$, and $a = 0.5$.

5. Print out graphs of the phase portrait showing the behavior of solutions near each equilibrium point (by using `zoom in`) in each of the three cases of the parameter a described in the previous problem. What is the bifurcation point for the parameter a?

Note: your lab report for Lab 5 should consist of your solutions to the prelab problems, your solutions to lab problems 1–5 (including any graphs printed out), and a brief, original, conclusions paragraph summarizing what you have learned.