Math 216 — Second Midterm
27 March, 2019

First 3 Letters of Last Name: __ __ __  UM Id#: ________________
Instructor: ______________________________  Section: ______________

1. Do not open this exam until you are told to do so.

2. This exam has 11 pages including this cover. There are 7 problems. Note that the problems
   are not of equal difficulty, so you may want to skip over and return to a problem on which
   you are stuck.

3. Do not separate the pages of this exam, other than the formula sheet at the end of the exam.
   If they do become separated, write your name on every page and point this out to your
   instructor when you hand in the exam.

4. Please read the instructions for each individual problem carefully. One of the skills being
   tested on this exam is your ability to interpret mathematical questions, so instructors will
   not answer questions about exam problems during the exam.

5. Show an appropriate amount of work (including appropriate explanation) for each problem,
   so that graders can see not only your answer but how you obtained it. Include units in your
   answer where that is appropriate.

6. You may use no aids (e.g., calculators or notecards) on this exam.

7. Turn off all cell phones, remove all headphones, and place any watch you are using
   on the desk in front of you.

8. Note that this exam is printed double-sided. There is a blank page at the end of the
   exam, should you need more space for your work on any problem.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
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1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, DO NOT use Laplace transforms.

   a. [7 points] Find the general solution to \(2u''(t) + 12u'(t) + 16u(t) = 12e^{4t}\).

   b. [8 points] Find the solution to the initial value problem \(y''(t) + 6y'(t) + 9y(t) = 3e^{-3t},\ y(0) = 0, \ y'(0) = 1.\)
2. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO use Laplace transforms**.

a. [7 points] Find the solution to the initial value problem \( y'' + 4y' + 20y = 0, \ y(0) = 1, \ y'(0) = 5. \)

b. [8 points] Find the solution to the initial value problem \( y'' + 3y' + 2y = e^{-t}, \ y(0) = 0, \ y'(0) = 0. \)
3. [14 points] Suppose that $L[y] = y'' + p(t)y' + q(t)y$. (Note that $L[y]$ here is a differential operator, not the Laplace transform $\mathcal{L}\{y\}$.)

a. [7 points] If $L[t^2] = 2 + 2tp(t) + t^2q(t) = 0$ and $L[t^2\ln(t)] = (2\ln(t) + 3) + (2t\ln(t) + t)p(t) + t^2\ln(t)q(t) = 0$, which, if any, of the following functions $y$ are solutions to $L[y] = 0$ on the domain $t > 0$? Which, if any, give a general solution on this domain? Why? (In these expressions, $c_1$ and $c_2$ are real constants.)

\begin{align*}
y_1 &= 5t^2 \\
y_2 &= 5t^2(1 + 2\ln(t)) \\
y_3 &= c_1t^2 + c_2t^2\ln(t) \\
y_4 &= -t^2\ln(t) \\
y_5 &= t^4\ln(t) \\
y_6 &= c_1t^2(1 + \ln(t)) \\
y_7 &= t^{-2}\ln(t) \\
y_8 &= W[t^2, t^2\ln(t)] = t^3 \\
y_9 &= c_1(5t^2 - 2c_2\ln(t))
\end{align*}

b. [7 points] Now suppose that $p(t) = 2$ and $q(t) = 10$, and let $L[y] = y'' + 2y' + 10y = g(t)$. For what $g(t)$ will the steady state solution to this problem be constant? Solve your equation with this $g(t)$ and explain how your solution confirms that your $g(t)$ is correct.
4. [15 points] In lab 2 we considered the van der Pol oscillator, modeled by the equation, \( x'' + \mu(x^2 - 1)x' + x = 0 \). Recall that there is a single critical point for this system, \( x = 0 \), near which we may model the behavior of the oscillator with the linear equation \( x'' - \mu x' + x = 0 \).

a. [5 points] Suppose that \( \mu = -1 \). Find the amplitude of the solution to the linear problem with initial condition \( x(0) = 2, x'(0) = 3 \). What is the time after which this amplitude never exceeds some value \( a_0 \)?

b. [5 points] Suppose we force the linear system with an oscillatory input, so that we are considering \( x'' - \mu x' + x = \cos(\omega t) \) (and \( \omega \neq 0 \)). For what values of \( \mu \) will the system have an oscillatory steady-state solution with frequency \( \omega \)?

c. [5 points] Suppose that, for some choice of \( \mu \), the system \( x'' - \mu x' + x = \cos(\omega t) \) has an oscillatory steady-state solution, and that the gain function \( G(\omega) \) for this solution is shown to the right, below. If the steady-state solution to the problem is \( y_{ss} = R \cos(t - \pi/2) \), what are the \( R \) in the solution, and \( \omega \) in the forcing term? Why?
5. [14 points] In each of the following we consider a linear, second order, constant coefficient operator $L$, so that $L[y] = 0$ is a homogenous differential equation. (Note, however, that the operator $L$ may be different in each of the parts below.) Let $y(0) = y_0$ and $y'(0) = v_0$, where $y_0$ and $v_0$ are real numbers.

a. [7 points] If the general solution to the equation $L[y] = 0$ is $y = c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t)$, what is the Laplace transform $Y(s) = L\{y(t)\}$?

b. [7 points] Now suppose that we are solving $L[y] = k$, for some constant $k$, and that $y_0$ and $v_0$ are both zero (so that $y(0) = y'(0) = 0$). If $Y(s) = L\{y(t)\}$ is $Y = \frac{k}{s(s+3)(s+4)}$, what is the differential equation we are solving, and the general solution to the complementary homogeneous problem? Explain how you know your answer is correct.
6. [15 points] Each of the following concerns a linear, second order, constant coefficient differential equation \( y'' + py' + qy = 0 \).

a. [7 points] If the general solution to the problem is \( y = c_1 e^{2t} + c_2 e^{4t} \), sketch a phase portrait for the system.

b. [8 points] Now suppose that for some real-valued \( \alpha \), we have \( p = 2\alpha \) and \( q = 1 \), so that we are considering \( y'' + 2\alpha y' + y = 0 \). For what values of \( \alpha \), if any
   (i) do all solutions to the differential equation decay to zero?
   (ii) are there solutions that do not decay to zero?
   (iii) will the general solution be a decaying sinusoidal function?
7. [12 points] For each of the following give an example, as indicated, and provide a short (one or two sentence) explanation of why your answer is correct.

a. [4 points] Give an example of an initial value problem with a linear, second-order, homogeneous differential equation for which there is no guarantee of a unique solution.

b. [4 points] Give an example of a linear, second-order, constant-coefficient, nonhomogeneous differential for which we cannot use the method of undetermined coefficients. What form will the general solution to your equation take?

c. [4 points] Give an example of a linear, second-order, nonhomogeneous differential equation for which the Laplace transform of the dependent variable $y$ could be $L\{y(t)\} = Y(s) = \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3}$. 
This page provided for additional work.
Formulas, Possibly Useful

- Some Taylor series, taken about \( x = 0 \):
  \[
  e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n.
  \]

  About \( x = 1 \): \( \ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x - 1)^n}{n} \).

- Some integration formulas:
  \[
  \int uv'\,dt = uv - \int u'v\,dt; \quad \int te^t\,dt = te^t - e^t + C, \quad \int t \sin(t)\,dt = t \sin(t) + \cos(t) + C, \quad \int t \cos(t)\,dt = -t \cos(t) + \sin(t) + C.
  \]

- Euler’s formula: \( e^{i\theta} = \cos \theta + i \sin \theta \).

- A coarse summary of partial fractions:
  \[
  \frac{1}{(s + r_1)(s + r_2)^2((s + h) + k^2)} = \frac{A}{s + r_1} + \frac{B}{s + r_2} + \frac{C}{(s + r_2)^2} + \frac{D(s + h) + E}{(s + h)^2 + k^2}.
  \]

Some Laplace Transforms

<table>
<thead>
<tr>
<th>( f(t) = \mathcal{L}^{-1}{F(s)} )</th>
<th>( F(s) = \mathcal{L}{f(t)} )</th>
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<tbody>
<tr>
<td>1. ( 1 )</td>
<td>( \frac{1}{s}, \ s &gt; 0 )</td>
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<tr>
<td>2. ( e^{at} )</td>
<td>( \frac{1}{s - a}, \ s &gt; a )</td>
</tr>
<tr>
<td>3. ( t^n )</td>
<td>( \frac{n!}{s^{n+1}} )</td>
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<tr>
<td>4. ( \sin(at) )</td>
<td>( \frac{a}{s^2 + a^2} )</td>
</tr>
<tr>
<td>5. ( \cos(at) )</td>
<td>( \frac{s}{s^2 + a^2} )</td>
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<tr>
<td>A. ( f'(t) )</td>
<td>( sF(s) - f(0) )</td>
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<tr>
<td>A.1 ( f''(t) )</td>
<td>( s^2F(s) - sf(0) - f'(0) )</td>
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<tr>
<td>A.2 ( f^{(n)}(t) )</td>
<td>( s^nF(s) - \cdots - f^{(n-1)}(0) )</td>
</tr>
<tr>
<td>B. ( t^n f(t) )</td>
<td>( (-1)^nF^{(n)}(s) )</td>
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<tr>
<td>C. ( e^{ct}f(t) )</td>
<td>( F(s - c) )</td>
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