Math 216 — Final Exam
26 April, 2019

First 3 Letters of Last Name: □□□  UM Id#: __________
Instructor: ________________  Section: __________

1. **Do not open this exam until you are told to do so.**

2. This exam has 13 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.

3. Do not separate the pages of this exam, other than the formula sheet at the end of the exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.

4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.

5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.

6. You may use no aids (e.g., calculators or notecards) on this exam.

7. **Turn off all cell phones, remove all headphones, and place any watch you are using on the desk in front of you.**

8. **Note that this exam is printed double-sided.** There is a blank page at the end of the exam, should you need more space for your work on any problem.

9. The last two problems on this exam are the “skills” questions for the test. **DO NOT COMPLETE THE LAST TWO PROBLEMS (ONLY) IF YOU HAVE COMPLETED THE MASTERY ASSESSMENT.** If you have not completed the mastery assessment, you should complete the last two problems.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
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<tbody>
<tr>
<td>1</td>
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</table>
1. [12 points] Six matrices and their eigenvalues and eigenvectors are given below. Use this information to answer the questions below. Be sure that you explain your answers.

<table>
<thead>
<tr>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>\begin{pmatrix} -1 &amp; 2 \ -1 &amp; -3 \end{pmatrix}</td>
<td>\begin{pmatrix} 2 &amp; 2 \ 1 &amp; 3 \end{pmatrix}</td>
<td>\begin{pmatrix} -2 &amp; 2 \ 1 &amp; -3 \end{pmatrix}</td>
<td>\begin{pmatrix} -1 &amp; 3 \ 2 &amp; -2 \end{pmatrix}</td>
<td>\begin{pmatrix} -2 &amp; -2 \ -1 &amp; -3 \end{pmatrix}</td>
<td>\begin{pmatrix} -3 &amp; -1 \ 1 &amp; -1 \end{pmatrix}</td>
</tr>
<tr>
<td>$\lambda_{1,2} = -2 \pm i$</td>
<td>$\lambda_{1,2} = 1, 4$</td>
<td>$\lambda_{1,2} = -4, -1$</td>
<td>$\lambda_{1,2} = -4, 1$</td>
<td>$\lambda_{1,2} = -4, -1$</td>
<td>$\lambda_{1,2} = -2, -2$</td>
</tr>
<tr>
<td>$v_1 = \begin{pmatrix} 2 \ -1 + i \end{pmatrix}$</td>
<td>$v_1 = \begin{pmatrix} -2 \ 1 \end{pmatrix}$</td>
<td>$v_1 = \begin{pmatrix} -1 \ 1 \end{pmatrix}$</td>
<td>$v_1 = \begin{pmatrix} -1 \ 1 \end{pmatrix}$</td>
<td>$v_1 = \begin{pmatrix} 1 \ 1 \end{pmatrix}$</td>
<td>$v_1 = \begin{pmatrix} -1 \ 1 \end{pmatrix}$</td>
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<tr>
<td>$v_2 = \begin{pmatrix} 2 \ -1 - i \end{pmatrix}$</td>
<td>$v_2 = \begin{pmatrix} 1 \ 1 \end{pmatrix}$</td>
<td>$v_2 = \begin{pmatrix} 2 \ 1 \end{pmatrix}$</td>
<td>$v_2 = \begin{pmatrix} 3 \ 2 \end{pmatrix}$</td>
<td>$v_2 = \begin{pmatrix} -2 \ 1 \end{pmatrix}$</td>
<td>$v_2 = \begin{pmatrix} 1 \ 0 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

**a. [6 points]** Write a linear system involving one of the $A_j$ that could have the phase portrait shown to the right.

**b. [6 points]** Write a linear system involving one of the $A_j$ that could have the phase portrait shown to the right.
2. [12 points] For each of the following, give an example, as indicated.

   a. [4 points] Give a first-order differential equation that could have the phase line shown to the right.

   b. [4 points] Give a second-order, linear, constant-coefficient, nonhomogeneous differential equation that could have the response shown to the right.

   c. [4 points] Give a system of two linear, first-order, constant-coefficient differential equations which have an isolated critical point at the origin that is an unstable saddle point.
3. [12 points] Suppose a model for a physical system (e.g., a circuit or a mass-spring system) is given by the differential equation $L[y] = y'' + ay' + by = k$ (where $a$, $b$, and $k$ are real numbers).

   a. [4 points] If the solution to the problem with some initial conditions is $y = e^{-t} \cos(2t) - e^{-t} \sin(2t) + 2$, what can you say about $a$, $b$, and $k$?

   b. [4 points] If the solution to the problem with some initial conditions is $y = e^{-t} \cos(2t) - e^{-t} \sin(2t) + 2$, sketch a phase portrait for the system. Be sure it is clear how you obtain your solution.

   c. [4 points] Now suppose that the solution to the problem with some initial conditions is $y = e^{-t} \cos(2t) - e^{-t} \sin(2t) + 2$, and that at some time $t = t_0$ we remove the forcing term $(k)$. Write a single differential equation you could solve to find $y$ for all $t \geq 0$. What initial conditions apply at $t = 0$?
4. [12 points] Consider the predator-prey model with harvesting (harvesting here implies hunting by humans, e.g., fishing if the populations are fish) given by

\[ x' = x(3 - x - y) - 2, \quad y' = y(-3 + x). \]

Note that as \( x \) and \( y \) are populations, we must have \( x, y \geq 0 \).

a. [3 points] Explain what each term in the equation for \( x \) models. Is \( x \) or \( y \) the predator? Which population is being harvested?

b. [7 points] By doing an appropriate linear analysis, sketch a phase portrait for this system.

c. [2 points] Based on your answer to (b), sketch what you expect the behavior of the solution to the system will be as a function of time if \( x(0) = 3 \) and \( y(0) = 1 \). How would you expect this to differ from the behavior with the initial condition \( x(0) = 1, y(0) = 1 \)?
5. [10 points] Consider the linear system

\[
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & 0 & \alpha^2 \\ 0 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

a. [5 points] For what values of \( \alpha \), if any, will all solutions to the system remain bounded as \( t \to \infty \)?

b. [5 points] Now suppose that \( \alpha = 2 \). Are there any initial conditions for which solutions to the system will remain bounded? If so, what are they? Explain.

\[ \text{Possibly useful: } \text{det} \left( \begin{pmatrix} a & 0 & b \\ 0 & c & d \\ e & 0 & f \end{pmatrix} \right) = acf - bce. \]
6. [12 points] In lab we considered an electrical system \( y'' + \omega_0^2 y = f(t) \) which produced a response similar to that shown in the figure to the right. In this problem, we will take \( \omega_0 = \pi \), and use the initial conditions \( y(0) = 0 \), \( y'(0) = 1 \).

   a. [6 points] If we pick \( f(t) = k\delta(t - t_0) \), what is \( t_0 \)?
   Solve the problem with this \( f(t) \) to find a value of \( k \) that results in a solution that could produce a graph similar to this one. Explain your logic.

   b. [6 points] Now suppose \( f(t) = kI(t) \), where \( I(t) \) is the finite-width impulse we used in lab 4. Let \( I(t) \) have height 8 and width \( \frac{1}{8} = 0.125 \), applied at \( t = 2 \) (that is, \( I(t) = 8 \) for \( 2 < t < 2.125 \), and is zero elsewhere). Solve the resulting problem to find \( y \). What should \( k \) be to produce a solution similar to that graphed above (your work in lab may be useful here)? If you used this value of \( k \) in (a), how would the response graph be different?
7. [15 points] DO complete this problem if you have NOT completed the mastery assessment. DO NOT complete it if you have completed the mastery assessment. Find explicit, real-valued solutions for each of the following.

a. [7 points] Find $Q(z)$ if $(z + 1)Q' = 3Q^2$, $Q(0) = 4$.

b. [8 points] Find the general solution to the first-order system $(x' y') = \begin{pmatrix} 0 & 13 \\ -1 & -6 \end{pmatrix} (x y)$. 
8. [15 points] DO complete this problem if you have NOT completed the mastery assessment. DO NOT complete it if you have completed the mastery assessment. Find explicit, real-valued solutions for each of the following.

a. [7 points] Find \( W(t) \) if \( W'' - 2W' - 8W = 24t + 54, \) \( W(0) = 0, \) \( W'(0) = 0. \)

b. [8 points] Find \( p(t) \) if \( p'' + 8p' + 16p = 6\delta(t - 2), \) \( p(0) = 0, \) \( p'(0) = 9. \)
This page provided for additional work.
Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  
  \[
  e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
  \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
  \]

  About $x = 1$: \( \ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \).

- Some integration formulas:
  \[
  \int uv' \, dt = uv - \int u'v \, dt; \quad \int te^t \, dt = te^t - e^t + C, \quad \int t \cos(t) \, dt = t \sin(t) + \cos(t) + C, \quad \int t \sin(t) \, dt = -t \cos(t) + \sin(t) + C.
  \]

- Euler’s formula: \( e^{i\theta} = \cos \theta + i \sin \theta \).

- A coarse summary of partial fractions:
  \[
  \frac{1}{(s + r_1)(s + r_2)^2((s + h)^2 + k^2)} = \frac{A}{s + r_1} + \frac{B}{s + r_2} + \frac{C}{(s + r_2)^2} + \frac{D(s + h) + E}{(s + h)^2 + k^2}.
  \]

Some Laplace Transforms

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) = \mathcal{L}{f(t)} )</th>
<th>( F(s) = \mathcal{L}{F(s)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( 1 )</td>
<td>( \frac{1}{s}, \ s &gt; 0 )</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>2. ( e^{at} )</td>
<td>( \frac{1}{s-a}, \ s &gt; a )</td>
<td>( \frac{e^{at}}{s-a} )</td>
</tr>
<tr>
<td>3. ( t^n )</td>
<td>( \frac{n!}{s^{n+1}} )</td>
<td>( \frac{n!}{s^{n+1}} )</td>
</tr>
<tr>
<td>4. ( \sin(at) )</td>
<td>( \frac{a}{s^2 + a^2} )</td>
<td>( \frac{a}{s^2 + a^2} )</td>
</tr>
<tr>
<td>5. ( \cos(at) )</td>
<td>( \frac{s}{s^2 + a^2} )</td>
<td>( \frac{s}{s^2 + a^2} )</td>
</tr>
<tr>
<td>6. ( u_c(t) )</td>
<td>( \frac{e^{-cs}}{s} )</td>
<td>( \frac{e^{-cs}}{s} )</td>
</tr>
<tr>
<td>7. ( \delta(t - c) )</td>
<td>( e^{-cs} )</td>
<td>( e^{-cs} )</td>
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</table>

A. \( f'(t) \) | \( sF(s) - f(0) \) | \( sF(s) \) |
A.1 \( f''(t) \) | \( s^2F(s) - sf(0) - f'(0) \) | \( s^2F(s) \) |
A.2 \( f^{(n)}(t) \) | \( s^nF(s) - \ldots - f^{(n-1)}(0) \) | \( s^nF(s) \) |
B. \( t^n f(t) \) | \( (-1)^n F^{(n)}(s) \) | \( (-1)^n F^{(n)}(s) \) |
C. \( e^{ct} f(t) \) | \( F(s-c) \) | \( F(s-c) \) |
D. \( u_c(t) f(t-c) \) | \( e^{-cs} F(s) \) | \( e^{-cs} F(s) \) |
E. \( f(t) \) (periodic with period \( T \)) | \( \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t) \, dt \) | \( \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t) \, dt \) |
F. \( \int_0^t f(x)g(t-x) \, dx \) | \( F(s)G(s) \) | \( F(s)G(s) \) |