Math 216–W19 Written Homework 2 Solutions

Instructions: Solve each of these problems. Your solution should be complete and written out in complete sentences. Where graphs are needed, you may include a print-out of output from Matlab (or another program, if you prefer).

1. Problem 33 in §3.1 of Brannan and Boyce (p.129 in the 3rd ed. text). Also complete parts (c) and (d), below.

(c) For what value of $\alpha$, if any, does $Ax = 0$ have an infinite number of solutions? Why? What are the eigenvalues of $A$ in this case? Explain why this makes sense.

(d) For what value of $\alpha$, if any, does $Ax = x$ have a solution? If this is possible, how many solutions are there? Why?

Solution note: (a) We find eigenvalues as expected, requiring
\[ \det \left( \begin{array}{cc} 2 - \lambda & \alpha \\ 1 & -3 - \lambda \end{array} \right) = \lambda^2 + \lambda - 6 - \alpha = 0. \] The quadratic formula gives $\lambda = -\frac{1}{2} \pm \frac{1}{2} \sqrt{25 + 4\alpha}$.

(b) The nature of the eigenvalues therefore depends on whether $25 + 4\alpha > 0$, $< 0$, or $= 0$. If $\alpha > -\frac{25}{4}$, both eigenvalues are real. If $\alpha = -\frac{25}{4}$, there is a single repeated eigenvalue, $\lambda = -\frac{1}{2}$. If $\alpha < -\frac{25}{4}$, there is a complex-conjugate pair of eigenvalues.

(c) $Ax = 0$ has an infinite number of solutions when $\det(A) = -6 - \alpha = 0$, or, when $\alpha = -6$. In this case eigenvalues of $A$ are given by $\det \left( \begin{array}{cc} 2 - \lambda & -6 \\ 1 & -3 - \lambda \end{array} \right) = \lambda^2 + \lambda = 0$, so $\lambda = 0$ or $\lambda = -1$. It makes sense that we should have a zero eigenvalue, because we’re requiring $\det(A) = \det(A - 0I) = 0$.

(d) For $Ax = x$, we must have $\lambda = 1$. This will happen when $\lambda = -\frac{1}{2} + \frac{1}{2} \sqrt{25 + 4\alpha} = 1$, or, $\sqrt{25 + 4\alpha} = 3$. Thus $\alpha = -4$. In this case there are an infinite number of solutions $x$, because $x$ is (a multiple of) an eigenvector.

2. Problem 30 in §3.2 of Brannan and Boyce (p.144 in the 3rd ed. text). For part (d), solve the problem by the following procedure:

i. Let $Q_1 = x + Q_1^E$ and $Q_2 = y + Q_2^E$, plug into the system to see that you obtain a homogeneous system $x' = Ax$. Note how this is the same as the procedure shown in §3.3, p.147.

ii. Solve this system to find $x = \begin{pmatrix} x \\ y \end{pmatrix}$ and then use this to write the solution for $Q_1$ and $Q_2$. 

Solution note: (a) To derive the governing equations, let $Q_1$ be the amount of salt in tank 1 (in oz), and similarly $Q_2$. Then $\frac{dQ_1}{dt} = (\text{rate of input}) - (\text{rate of output})$. For $Q_1$, input is 1.5 gal/min of solution at 1 oz/gal, plus 1.5 gal/min of the solution in tank 2, which has a concentration of $Q_2/20$ oz/gal. The output from tank 1 is 3 gal/min with a concentration of $Q_1/30$ oz/gal. Thus the equation for $Q_1$ is

$$\frac{dQ_1}{dt} = 1.5 + 1.5 \frac{Q_2}{20} - 3 \frac{Q_1}{30} = -0.1Q_1 + 0.075Q_2 + 1.5.$$  

Similarly, for tank 2, input is 1 gal/min at 3 oz/gal and 3 gal/min with a concentration of $Q_1/30$. Output is 4 gal/min with a concentration of $Q_2/20$, so

$$\frac{dQ_2}{dt} = 3 + 3 \frac{Q_1}{30} - 4 \frac{Q_2}{20} = 0.1Q_1 - 0.2Q_2 + 3.$$  

The initial conditions are given by the problem, $Q_1(0) = 55$ and $Q_2(0) = 26$.

(b) In matrix notation, with $A = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix}$, $b = \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$, and $x = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, we have $x' = Ax + b$.

(c) Equilibrium values are when $Q_1' = Q_2' = 0$. Adding the two equations together, we have $-0.125Q_2 + 4.5 = 0$, so that $Q_2^E = 36$. Solving either equation for $Q_1$ then gives $Q_1^E = 42$.

(d) Letting $Q_1 = x + Q_1^E$ and $Q_2 = y + Q_2^E$ and plugging in, we have $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, with $x(0) = 13$ and $y(0) = -10$. Eigenvalues are given by $(-0.1-\lambda)(-0.2-\lambda)-0.0075 = \lambda^2 + 0.3\lambda + 0.0125 = 0$. Solving with the quadratic formula gives $\lambda = -0.25$ and $\lambda = -0.05$. If $\lambda = -0.25$, the components of the eigenvector satisfy $0.1v_1 + 0.05v_2 = 0$, so $v = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. If $\lambda = -0.05$, we have $0.1v_1 - 0.15v_2 = 0$, so that $v = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Thus

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-0.25t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-0.05t} + \begin{pmatrix} 42 \\ 36 \end{pmatrix}.$$  

Applying the initial conditions, we find $c_1 = -7$, $c_2 = 2$. Plotting the components, we have the graph below.
(e) The corresponding phase portrait, centered on the equilibrium point, is given below.

3. Problem 20 in §3.3 of Brannan and Boyce (p.166 in the 3rd ed. text). Also complete part (d) below.

(d) Find a matrix $A$ that could give the eigenvalues and eigenvectors for this system.
Solution note: (a),(b) The phase portrait and trajectory are shown below. The dashed trajectory is that through (2, 3).

(c) The component plots are below.

(d) We can solve for the matrix \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) by using the eigenvalue/eigenvector pairs: we know that \( A\mathbf{v} = \lambda\mathbf{v} \). Using both eigenvalue/eigenvector pairs, we get the system of equations

\[
\begin{align*}
    a_{11} + 2a_{12} &= 1, & a_{21} + 2a_{22} &= 2 \\
    a_{11} - 2a_{12} &= 2, & a_{21} - 2a_{22} &= -4.
\end{align*}
\]

Solving, we find \( a_{11} = \frac{3}{2}, \ a_{12} = -\frac{1}{4}, \ a_{21} = -1, \ a_{22} = \frac{3}{2} \). That is, \( A = \begin{pmatrix} \frac{3}{2} & -\frac{1}{4} \\ -1 & \frac{3}{2} \end{pmatrix} \).

4. Note that the linearized van der Pol system we consider in lab 2 (equation (5) in the prelab) is of the form of Problem 14 in §3.4 of Brannan and Boyce (p.177 in the 3rd ed. text). Follow the instructions for that problem and the linearized van der Pol system.
Solution note: (a) The eigenvalues are given by \( \det \left( \begin{array}{cc} -\lambda & 1 \\ -1 & \mu - \lambda \end{array} \right) = \lambda^2 - \mu \lambda + 1 = 0 \), so that \( \lambda = \frac{1}{2} \mu \pm \frac{1}{2} \sqrt{\mu^2 - 4} \).

(b) Considering only positive values of \( \mu \), the critical value of \( \mu \) is \( \mu = 2 \), where the square root goes from negative to positive.

(c) Just below the critical value of \( \mu \) we have a spiral source, as shown in the figure to the left, below. We can determine the direction of the spiral by noting that at (0,1) trajectories have \( (x', y') = (1, \mu) \), that is, to the right and up. Just above the critical value of \( \mu = 2 \) we expect two straight line solutions but with behavior similar to that shown in the spiral. With \( \mu = 2.05 \), eigenvalues are \( \lambda = 1.25 \) and 0.8, with \( \mathbf{v} = (4 \ 5)^T \) and \( \mathbf{v} = (5 \ 4)^T \), so that, we have a nodal source, as shown in the figure to the right.