Math 216–W19 Written Homework 5 Solutions

Instructions: Solve each of these problems. Your solution should be complete and written out in complete sentences. Where graphs are needed, you may include a print-out of output from Matlab (or another program, if you prefer).

1. Problem 14 in §5.7 of Brannan and Boyce (p.350 in 3rd ed. text). Also complete part (e) if take $\gamma = \frac{1}{2}$ and replace the forcing term, $\delta(t-1)$, with a sum $\sum_{k=1}^{N} \delta(t - kd)$, what $d$ should we pick to maximize the amplitude of the response? Solve the problem with this $d$ and plot your solution for $N = 5$.

Solution note: (a) With $\gamma = \frac{1}{2}$, we have, on transforming the equation and taking $Y(s) = \mathcal{L}\{y(t)\}$, $(s^2 + \frac{1}{2}s + 1)Y(s) = e^{-s}$, so that $Y = \frac{e^{-s}}{(s+\frac{1}{2})^2 + \frac{1}{16}}$. Thus

$$y = \frac{4}{\sqrt{15}} u_1(t)e^{-(t-1)/4} \sin\left(\frac{\sqrt{15}}{4}(t-1)\right).$$

(b) This attains its maximum when the sine reaches its first maximum, which is when $\frac{\sqrt{15}}{4}(t - 1) = \frac{\pi}{2}$, or, $t_1 = \frac{2\pi}{\sqrt{15}} + 1 \approx 2.662$, at which point $y = e^{-(t_1-1)/4} \approx 0.667$. (Technically, to find the precise maximum we should take the derivative and set it to zero, which would find the $t$ value at which the precise maximum is attained; this gives $t \approx 2.361$, $y \approx 0.712$. The method here ignores the effect of the exponential decay, which is mostly but not entirely correct.)

(c) If $\gamma = \frac{1}{4}$, we have $Y = \frac{e^{-s}}{(s+\frac{1}{4})^2 + \frac{1}{64}}$, and

$$y = \frac{8}{\sqrt{63}} u_1(t)e^{-(t-1)/8} \sin\left(\frac{\sqrt{63}}{8}(t-1)\right).$$

The maximum now occurs when $t = t_2 = \frac{4\pi}{\sqrt{63}} + 1 \approx 2.583$, where $y = e^{-(t_2-1)/8} \approx 0.673$. (Similarly, taking derivatives, we have the more precise value $t \approx 2.457$, $y \approx 0.834$.)

(d) To see what happens as $\gamma$ decreases let $\gamma = \frac{1}{g}$ and note that the $t$ value at the maximum is $t = t_m = \frac{2\pi}{2\sqrt{g^2-1}} + 1$. As $g \to \infty$, this will converge to $t = \frac{\pi}{2} + 1$. The value there will be one, the value of the sine at $\pi/2$. 

Note that the complementary homogenous solution to the problem is \( y_c = c_1 e^{-t/4} \cos \left( \frac{\sqrt{15}}{4} t \right) + c_1 e^{-t/4} \sin \left( \frac{\sqrt{15}}{4} t \right) \). To maximize the response, we want to trigger the impulses at the end of successive periods of the homogeneous solution. The period of the solution is \( T = \frac{8\pi}{\sqrt{15}} \), so let \( d = T = \frac{8\pi}{\sqrt{15}} \). We can read the resulting solution from our work in (a):

\[
y = \sum_{k=1}^{N} \frac{4}{\sqrt{15}} u_k T(t) e^{-t/4} \sin \left( \frac{\sqrt{15}}{4} (t - kT) \right).
\]

The resulting graph is shown below.

Numerically finding the maximum amplitude, we find it to be \( y \approx 0.886 \); empirically testing our choice of forcing, we can similarly calculate the value if we force with \( \sum_{k=1}^{N} \delta(t - k(T \pm \epsilon)) \). Picking \( \epsilon = 0.25 \), we find the maximum is then \( (+\epsilon) y \approx 0.879 \) or \( (-\epsilon) y \approx 0.878 \).

2. Problem 4 in §6.4 of Brannan and Boyce (p.419 in 3rd ed. text). Also complete part (a) are there any initial conditions for which all solutions will have no oscillatory component? If so, what are they?

**Solution note:** Finding eigenvalues, we require (in the following we calculate the determinant by expanding along the bottom row)

\[
\begin{vmatrix}
-4 - \lambda & 2 & -1 \\
-6 & -\lambda & -3 \\
0 & 8/3 & -2
\end{vmatrix} = \frac{8}{3}((-4 - \lambda)(-3) - (-6)(-1)) + (-2 - \lambda)((-4 - \lambda)(-\lambda) - (-6)(2))
\]

\[
= -8(\lambda + 4) - (\lambda + 2)(\lambda^2 + 4\lambda + 12)
\]

\[
= -(\lambda + 2)(\lambda^2 + 4\lambda + 20)
\]

\[
= -(\lambda + 2)((\lambda + 2)^2 + 16) = 0.
\]

Thus \( \lambda = -2 \) or \( \lambda = -2 \pm 4i \). If \( \lambda = -2 \), the eigenvector satisfies
\[
\begin{pmatrix}
-2 & 2 & -1 \\
-6 & 2 & -3 \\
0 & 8/3 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\]
so that \(v_2 = 0\) and \(v_3 = -2v_1\). We can take \(v = \begin{pmatrix}1 \\ 0 \\ -2\end{pmatrix}\). Similarly, if \(\lambda = -2 + 4i\), we have
\[
\begin{pmatrix}
-2 - 4i & 2 & -1 \\
-6 & 2 - 4i & -3 \\
0 & 8/3 & -4i
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

The last equation gives \(v_2 = \frac{3}{2}iv_3\), so we may take \(v_3 = 2\) so that \(v_2 = 3i\). Then the second equation gives \(-6v_1 = (-2 + 4i)(3i) + 3(2) = -6i - 12 + 6 = -6 - 6i\), and \(v_1 = 1 + i\). (Note that then the first equation becomes \((-2 - 4i)(1 + i) + 2(3i) - 2 = -2 + 4 - 2i + 6i - 2 = 0\).) So the eigenvector is \(v = \begin{pmatrix}1 + i \\ 3i \\ 2\end{pmatrix}\). Separating real and imaginary parts of the resulting solution and combining with the real-valued solution, we have the general solution
\[
x = c_1 \begin{pmatrix}1 \\ 0 \\ -2\end{pmatrix} e^{-2t} + c_2 \begin{pmatrix}\cos(4t) - \sin(4t) \\ -3\sin(4t) \\ 2\cos(4t)\end{pmatrix} e^{-2t} \\
+ c_3 \begin{pmatrix}\cos(4t) + \sin(4t) \\ 3\cos(4t) \\ 2\sin(4t)\end{pmatrix} e^{-2t}.
\]

(a) Any solutions that exclude the second two solutions in the general solution will not be oscillatory. That is, any solution that lies on the line determined by the first eigenvector: thus, if we require \(x(0) = k \begin{pmatrix}1 \\ 0 \\ -2\end{pmatrix}\) we will have strictly exponentially decaying solutions with no oscillatory component.

3. In lab 5 we consider the Lorenz equations
\[
x' = \sigma(-x + y) \\
y' = r x - y - xz \\
z' = -b z + xy
\]
In the following, take \(\sigma = 10\) and \(b = \frac{8}{3}\).

(a) Find all of the critical points for the Lorenz system in terms of the
For what values of $r$ is there only one critical point? More than one?

**Solution note:** We have $0 = \sigma(-x + y)$, $0 = rx - y - xz$, and $0 = -bz + xy$. Note that $(x, y, z) = (0, 0, 0)$ is one solution. To find other solutions, note that the first gives $x = y$, so the latter two become

\begin{align*}
0 &= x(r - 1 - z) \\
0 &= -bz + x^2.
\end{align*}

From these, we have $x = y = z = 0$, or $z = r - 1$, so that $x = y = \pm \sqrt{b(r - 1)}$. With $b = 8/3$, this is

$$(x, y, z) = (\pm \sqrt{\frac{8}{3}(r - 1)}, \pm \sqrt{\frac{8}{3}(r - 1)}, r - 1).$$

These exist only for $r > 1$, so for $r \leq 1$ there is only one critical point, $(0, 0, 0)$, and for $r > 1$ there are three.

(b) Find the Jacobian for the Lorenz system.

**Solution note:** The Jacobian is

\[
J = \begin{pmatrix}
-\sigma & \sigma & 0 \\
r - z & -1 & -x \\
y & x & -b
\end{pmatrix} = \begin{pmatrix}
-10 & 10 & 0 \\
r - z & -1 & -x \\
y & x & -\frac{8}{3}
\end{pmatrix}.
\]

(c) Use your work from (b) to find a linear system approximating the Lorenz system at the critical point $(0, 0, 0)$. Find the eigenvalues of the coefficient matrix of the system and determine the values of $r$ for which the critical point is stable and the values for which it is unstable (consider $r > 0$ only).

**Solution note:** At $(0, 0, 0)$, our linear approximation is $u' = J(0, 0, 0)u$, or

\[
u' = \begin{pmatrix}
-10 & 10 & 0 \\
r & -1 & 0 \\
0 & 0 & -\frac{8}{3}
\end{pmatrix} u.
\]

The eigenvalues of the coefficient matrix are given by (expanding along the bottom row)
\[
\begin{vmatrix}
-10 - \lambda & 10 & 0 \\
r & -1 - \lambda & 0 \\
0 & 0 & -8/3 - \lambda
\end{vmatrix}
\]
\[
= \left(-\frac{8}{3} - \lambda\right)((-10 - \lambda)(-1 - \lambda) - 10r)
\]
\[
= -(\lambda + \frac{8}{3})(\lambda^2 + 11\lambda + 10(1 - r)) = 0.
\]
Thus we have \(\lambda = -\frac{8}{3}\), or \(\lambda^2 + 11\lambda + 10(1 - r) = 0\). Applying the quadratic formula, the latter gives
\[
\lambda = -\frac{11}{2} \pm \frac{1}{2} \sqrt{11^2 - 40(1 - r)}.
\]
This will give a positive root when \(1 - r < 0\), that is, when \(r > 1\). Thus the critical point is stable for \(0 < r < 1\), and unstable for \(r > 1\).

4. Problem 24 in §7.2 of Brannan and Boyce (p.476 in 3rd ed. text). Also complete part (d) sketch a phase portrait for the system given your linear analysis.

**Solution note:**
(a) Clearly \((0, 0)\) is a critical point of each system. To determine the corresponding linear system, we find the Jacobian for each. For (i),
\[
J(0, 0) = \begin{vmatrix}
3x^2 + y^2 & 1 + 2xy \\
-1 + 2xy & x^2 + 3y^2
\end{vmatrix}_{(x, y) = (0, 0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
and for (ii),
\[
J(0, 0) = \begin{vmatrix}
-3x^2 - y^2 & 1 - 2xy \\
-1 - 2xy & -x^2 - 3y^2
\end{vmatrix}_{(x, y) = (0, 0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
The eigenvalues of both are \(\lambda = \pm i\), so \((0, 0)\) is a center for both linear systems.

(b) From the note at the top of p.470, we know that if the right-hand sides of the nonlinear systems are twice differentiable, the systems are almost linear. In this case the right-hand sides are polynomials and thus infinitely differentiable, so both systems are almost linear. Alternately, we can calculate an appropriate limit (equation (5) on p.468). For system (i), we have
\[
g(x) = f(x) - Ax = \begin{pmatrix}
y + x(y^2 + y^2) - y \\
-x + y(x^2 + y^2) - (-x)
\end{pmatrix} = \begin{pmatrix} x(x^2 + y^2) \\ y(x^2 + y^2) \end{pmatrix},
\]
so that
\[
\lim_{x \to 0} \frac{\|g(x)\|}{\|x\|} = \lim_{x \to 0} \frac{\sqrt{(x^2 + y^2)(x^2 + y^2)^2}}{\sqrt{x^2 + y^2}} = \lim_{x \to 0} (x^2 + y^2) = 0.
\]
Similarly, for system (ii),
\[
g(x) = f(x) - Ax = \begin{pmatrix} y - x(y^2 + y^2) - y \\ -x - y(x^2 + y^2) - (-x) \end{pmatrix} = \begin{pmatrix} -x(x^2 + y^2) \\ -y(x^2 + y^2) \end{pmatrix},
\]
and the limit calculation proceeds identically.

(c) We can do the analysis in terms of \( r \) by noting that systems (i) and (ii) are given by \( x' = y \pm x(x^2 + y^2) \), \( y' = -x \pm y(x^2 + y^2) \) (with the + taken for (i) and – for (ii)). Taking \( x \) times the first equation and \( y \) times the second and adding the two equations, we have
\[
dxdt + dydt = xy \pm x^2(x^2 + y^2) - xy \pm y^2(x^2 + y^2) = \pm(x^2 + y^2)^2.
\]
With \( r \frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} \) and \( r^2 = x^2 + y^2 \), this is \( rr' = \pm r^4 \). (Thus, for system (ii), \( \frac{dr}{dt} < 0 \).) Solving, we have \( r^{-3}r' = \pm 1 \), so that \( -\frac{1}{2}r^{-2} = \pm t + C' \), and \( r = \frac{1}{\sqrt{t + C}} \) (taking \( C = -2C' \)). Then with \( r(0) = r_0 \), \( C = 1/r_0^2 \).

For (i), we have \( r = (\frac{1}{r_0^2} - 2t)^{-1/2} \), and for (ii), \( r = (\frac{1}{r_0^2} + 2t)^{-1/2} \). Thus for (ii), as \( t \to \infty \), \( r \to 0 \). For (i), as \( t \to \frac{1}{r_0^2} \), \( r \to \infty \).

(d) The linear phase portrait for either system is just a center, shown below.