Solutions to the First Midterm Exam

Problem 1.

a) Differentiating, we get

$$y' = -x^{-2} + x^{-1} \quad \text{and} \quad y'' = 2x^{-3} - x^{-2}.$$

Substituting that into the equation, we obtain

$$\begin{aligned} x^2 y'' + 2xy' - xy =& x^2\left(2x^{-3} - x^{-2}\right) + 2x\left(-x^{-2} + x^{-1}\right) - x\left(x^{-1} + \ln x\right) \\ =& 2x^{-1} - 1 - 2x^{-1} + 2 - 1 - x\ln x = -x\ln x, \end{aligned}$$

as required.

b) The general solution of the equation is

$$y = \int \left(\sin x \cos^2 x + x^{-2}\right)\, dx = \int \sin x \cos^2 x\, dx + \int x^{-2}\, dx = -\frac{\cos^3 x}{3} - x^{-1} + C.$$

We have

$$y(2\pi) = \frac{-\cos^3 2\pi}{3} - \frac{1}{2\pi} + C = -\frac{1}{3} - \frac{1}{2\pi} + C,$$

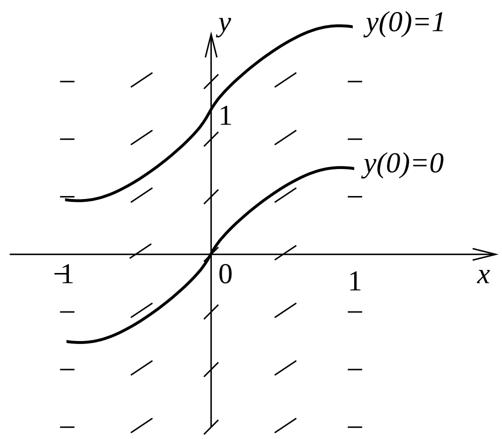
from which

$$C = \frac{4}{3} + \frac{1}{2\pi}.$$

Answer:

$$y = -\frac{\cos^3 x}{3} - \frac{1}{x} + \frac{4}{3} + \frac{1}{2\pi}.$$

c) We observe that $\sqrt{1-x^2}$ is defined for $-1 \leq x \leq 1$. The slope depends on the x coordinate only, always non-negative, grows from 0 at $x=-1$ to 1 at $x=0$ and then decays from 1 at $x=0$ to 0 at $x=1$. Hence the solutions will be increasing, concave up for $-1 \leq x \leq 0$ and then concave down for $0 \leq x \leq 1$.



Problem 2.

a) Separating the variables, we get

$$\frac{dy}{1-2y} = xe^{2x}\ dx \Longrightarrow \int \frac{dy}{1-2y} = \int xe^{2x}\ dx.$$

Integrating,

$$\int \frac{dy}{1-2y} = -\frac{1}{2}\ln|1-2y| + C$$

and

$$\int xe^{2x}\ dx = \frac{1}{2}\int x\ de^{2x} = \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\ dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

Hence we get the equation

$$-\frac{1}{2}\ln|1-2y| = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C, \quad \text{that is} \quad \ln|1-2y| = -xe^{2x} + \frac{e^{2x}}{2} + C.$$

Hence

$$1-2y = Ce^{-xe^{2x}+e^{2x}/2} \quad \text{and} \quad y = \frac{1}{2} - Ce^{-xe^{2x}+e^{2x}/2},$$

where "C" denotes an arbitrary constant. Note that we did not lose the solution $y = 1/2$ as it can be obtained by choosing $C = 0$.

Now,

$$y(0) = \frac{1}{2} - Ce^{1/2}, \quad \text{so} \quad C = -\frac{3}{2}e^{-1/2}.$$

Answer:

$$y = \frac{1}{2} + \frac{3}{2}e^{-xe^{2x}+e^{2x}/2-1/2}.$$

b) Writing

$$(x^2+1)\frac{dy}{dx} = x(1-3y),$$

separating the variables and integrating, we get

$$\int \frac{dy}{1-3y} = \int \frac{x}{(x^2+1)}\ dx.$$

That is,

$$-\frac{1}{3}\ln|1-3y| = \frac{1}{2}\ln(x^2+1) + C \quad \text{and} \quad \ln|1-3y| = -\frac{3}{2}\ln(x^2+1) + C.$$

Solving for y we obtain

$$1-3y = Ce^{-\frac{3}{2}\ln(x^2+1)} = C(x^2+1)^{-3/2} \quad \text{and} \quad y = \frac{1}{3} - \frac{1}{3}C(x^2+1)^{-3/2}.$$

Now,

$$y(0) = \frac{1}{3} - C \quad \text{and hence} \quad C = \frac{1}{3}.$$

Answer:

$$y = \frac{1}{3} - \frac{1}{3}(x^2+1)^{-3/2}.$$

c) We get the equation

$$\frac{dP}{dt} = kP^{1/3}$$

for some constant k. Noticing that $P'(0) = 100$ and $P(0) = 1000$ we get the equation for k

$$100 = k1000^{1/3} = 10k \quad \text{and hence} \quad k = 10.$$

Separating the variables and integrating, we get

$$\int \frac{dP}{P^{1/3}} = \int 10\ dt = 10t + C.$$

Therefore,

$$\frac{3}{2}P^{2/3} = 10t + C \quad \text{and} \quad P = \left(\frac{20}{3}t + C\right)^{3/2}.$$

Since $P(0) = 1000$, we must have $C^{3/2} = 1000$ and $C = 100$.

Therefore,

$$P(10) = \left(\frac{200}{3} + 100\right)^{3/2} = \left(\frac{500}{3}\right)^{3/2}.$$

Answer: The population at time $t = 10$ is $\left(\frac{500}{3}\right)^{3/2}$.

Problem 3.

a) We find the equilibrium solutions by solving the equation

$$-5x + 2x^3 = 0 \Longrightarrow x(2x^2 - 5) = 0$$

and so $x = 0$ or $x = \sqrt{5/2}$ or $x = -\sqrt{5/2}$.

Now, if the initial value $x(t_0) = x_0$ is slightly bigger than $\sqrt{5/2}$ then the slope dx/dt at $t = t_0$ is $x_0(2x_0^2 - 5) > 0$ and so $x(t)$ grows even further. If $x(t_0) = x_0$ is slightly smaller than $\sqrt{5/2}$ then the slope is $x_0(2x_0^2 - 5) < 0$ and so $x(t)$ becomes even smaller. This shows that $x(t) = \sqrt{5/2}$ is an unstable solution.

If the initial value $x(t_0) = x_0$ is slightly bigger than 0 then the slope dx/dt at $t = t_0$ is $x_0(2x_0^2 - 5) < 0$ and so $x(t)$ becomes smaller. If $x(t_0) = x_0$ is slightly smaller than 0 then the slope is $x_0(2x_0^2 - 5) > 0$ and so $x(t)$ grows. This shows that $x(t) = 0$ is a stable solution.

Finally, if the initial value $x(t_0) = x_0$ is slightly bigger than $-\sqrt{5/2}$ then the slope dx/dt at $t = t_0$ is $x_0(2x_0^2 - 5) > 0$ and so $x(t)$ grows even further. If $x(t_0) = x_0$ is slightly smaller than $-\sqrt{5/2}$ then the slope dx/dt at $t = t_0$ is $x_0(2x_0^2 - 5) < 0$ so $x(t)$ becomes even smaller. This shows that $x(t) = -\sqrt{5/2}$ is an unstable solution.

Answer: The equilibrium solutions are $x = \sqrt{5/2}$ (unstable), $x = 0$ (stable), and $x = -\sqrt{5/2}$ (unstable).

b) We have the equation

$$\frac{dv}{dt} = -kv \quad \text{with the additional constraints} \quad v(0) = 100 \quad \text{and} \quad v(20) = 10.$$

Solving the equation, we get the general solution $v(t) = Ce^{-kt}$. Using that $v(0) = 100$ we conclude that $C = 100$ and using that $v(20) = 10$ we conclude that

$$100e^{-20k} = 10 \quad \text{from which} \quad e^{-20k} = 0.1 \quad \text{and} \quad k = -\frac{\ln(0.1)}{20} = \frac{\ln 10}{20}.$$

The total distance travelled is

$$\int_0^{+\infty} v(t)\, dt = \int_0^{+\infty} Ce^{-kt}\, dt = \frac{C}{-k}e^{-kt}\Big|_{t=0}^{t=+\infty} = \frac{C}{k} = \frac{2000}{\ln 10}.$$

Answer: The boat will coast for $2000/\ln(10)$ meters.

c) First, we estimate $y(1.5)$.

We have $y(1) = 0$ and the slope at $x = 1$ equal to $k = 1^2 + 0^3 = 1$, so the estimate is

$$y(1.5) = y(1) + \frac{1}{2}k = 0 + \frac{1}{2} = \frac{1}{2}.$$

Next, we estimate $y(2)$.
We have the estimate $y(1.5) = 1/2$ and the slope at $x = 1.5$ is equal to

$$k = \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^3 = \frac{9}{4} + \frac{1}{8} = \frac{19}{8}.$$

Finally, we estimate $y(2)$:

$$y(2) = y(1.5) + \frac{1}{2}k = \frac{1}{2} + \frac{19}{16} = \frac{27}{16} = 1.6875.$$

Answer: The estimate is $y(2) = 27/16$.

Problem 4.

a) We look for solutions in the form $y = e^{rx}$ for some number r. Substituting such a y into the equation, we get $r^2 - 3r - 10 = 0$, from which $r = -2$ or $r = 5$. Hence the general solution is

$$y = C_1 e^{-2x} + C_2 e^{5x} \quad \text{for some numbers} \quad C_1, C_2.$$

Since $y(0) = 3$ we get $C_1 + C_2 = 3$ and since $y'(0) = -1$ we get $-2C_1 + 5C_2 = -1$. Hence we get the system

$$\begin{aligned} C_1 + C_2 &= 3 \\ -2C_1 + 5C_2 &= -1, \end{aligned}$$

from which $C_2 = 5/7$ and $C_1 = 16/7$.

Answer:

$$y = \frac{16}{7} e^{-2x} + \frac{5}{7} e^{5x}.$$

b) We get

$$z = \frac{2 \pm \sqrt{4 - 4 \cdot 5 \cdot 10}}{10} = \frac{2 \pm \sqrt{-196}}{10} = \frac{2 \pm 14i}{10} = \frac{1 \pm 7i}{5}.$$

Answer: The solutions are

$$\frac{1}{5} + \frac{7}{5} i \quad \text{and} \quad \frac{1}{5} - \frac{7}{5} i.$$

c) We look for solutions in the form $y = e^{rx}$ for some number r. Substituting such a y into the equation, we get

$$9r^2 + 6r + 25 = 0 \quad \text{or} \quad (3r + 1)^2 + 24 = 0.$$

Now,

$$(3r + 1)^2 = -24 \quad \text{and} \quad 3r + 1 = \pm i\sqrt{24}, \quad \text{so} \quad r = -\frac{1}{3} - \frac{\sqrt{24}}{3} i \quad \text{or} \quad r = -\frac{1}{3} + \frac{\sqrt{24}}{3} i.$$

Thus the general solution is

$$y = C_1 e^{-\frac{1}{3}x} \sin \frac{\sqrt{24}}{3} x + C_2 e^{-\frac{1}{3}x} \cos \frac{\sqrt{24}}{3} x.$$

Now, $y(0) = 2$ which gives us $C_2 = 2$.

Furthermore,

$$\begin{aligned} y' = &- \frac{1}{3}C_1 e^{-\frac{1}{3}x} \sin \sqrt{\frac{24}{3}}x + C_1 e^{-\frac{1}{3}x} \frac{\sqrt{24}}{3} \cos \frac{\sqrt{24}}{3}x \\ &- \frac{1}{3}C_2 e^{-\frac{1}{3}x} \cos \frac{\sqrt{24}}{3}x - \frac{\sqrt{24}}{3}C_2 e^{-\frac{1}{3}x} \sin \frac{\sqrt{24}}{3}x. \end{aligned}$$

From $y'(0) = 1$ we get

$$C_1 \frac{\sqrt{24}}{3} - \frac{1}{3}C_2 = 1, \quad \text{so} \quad C_1 = \frac{3}{\sqrt{24}} \left(1 + \frac{2}{3}\right) = \frac{5}{\sqrt{24}}.$$

Answer:

$$y = \frac{5}{\sqrt{24}} e^{-\frac{1}{3}x} \sin \frac{\sqrt{24}}{3}x + 2e^{-\frac{1}{3}x} \cos \frac{\sqrt{24}}{3}x.$$