Solutions to the Second Midterm Exam

Problem 1.

a) We look for a solution in the form $y = ae^{2x}$ for some a. Substituting y into the equation, we get

$$4ae^{2x} + 2ae^{2x} + ae^{2x} = e^{2x},$$

from which $7a = 1$ and $a = 1/7$.

Answer: A particular solution is

$$y = \frac{e^{2x}}{7}.$$

b) First, we look for a particular solution in the form $y = ae^{2x}$ for some a. Thus we get $y'' = 4ae^{2x}$ and the equation becomes $4ae^{2x} - 4ae^{2x} = e^{2x}$, which is a signature equation of duplication (resonance). Hence we try $y = axe^{2x}$ instead, so $y' = e^{2x}(a + 2ax)$ and $y'' = e^{2x}(2a + 4ax) + 2ae^{2x} = e^{2x}(4a + 4ax)$.

Substituting that into the equation, we get

$$e^{2x}(4a + 4ax) - 4axe^{2x} = 4ae^{2x} = e^{2x},$$

from which $a = 1/4$.

Answer: A particular solution is

$$y = \frac{xe^{2x}}{4}.$$

c) First, we solve the homogeneous equation

$$y'' - y = 0.$$

Looking for the solution in the form $y = e^{rx}$ we get $r^2 - 1 = 0$, from which the general solution is

$$y = C_1e^x + C_2e^{-x}$$

for some constants C_1 and C_2.

Now, we look for the solution of the original equation in the form

$$y = C_1(x)e^x + C_2(x)e^{-x}.$$

Differentiating, we get

$$y' = C_1'e^x + C_1e^x + C_2'e^{-x} - C_2e^{-x},$$

which, after the imposing the condition

$$C_1' e^x + C_2' e^{-x} = 0 \tag{1.1}$$

becomes just

$$y' = C_1 e^x - C_2 e^{-x}.$$

Differentiating further, we get

$$y'' = C_1' e^x + C_1 e^x - C_2' e^{-x} + C_2 e^{-x}.$$

Hence

$$y'' - y = C_1' e^x - C_2' e^{-x} = xe^{2x}. \tag{1.2}$$

Now we solve (1.1) and (1.2) for C_1' and C_2'.

If we add (1.1) and (1.2), we get $2C_1' e^x = xe^{2x}$, so $C_1' = xe^x/2$. Similarly, if we subtract (1.2) from (1.1), we get $2C_2' e^{-x} = -xe^{2x}$, so $C_2' = -xe^{3x}/2$.

Integrating by parts, we get

$$C_1(x) = \frac{1}{2}\int xe^x \, dx = \frac{xe^x}{2} - \frac{e^x}{2} + \text{constant}$$

and

$$C_2(x) = -\frac{1}{2}\int xe^{3x} \, dx = -\frac{xe^{3x}}{6} + \frac{e^{3x}}{18} + \text{some other constant.}$$

Since we are interested in particular solutions, we can choose the constants equal to 0 and get

$$y = C_1(x)e^x + C_2(x)e^{-x} = \frac{xe^{2x}}{2} - \frac{e^{2x}}{2} - \frac{xe^{2x}}{6} + \frac{e^{2x}}{18} = \frac{xe^{2x}}{3} - \frac{4e^{2x}}{9}.$$

Answer: A particular solution is

$$y(x) = e^{2x}\left(\frac{x}{3} - \frac{4}{9}\right).$$

d) From the general equation we have $mx'' + kx = 0$, where $m = 10$ and k is determined from the equation $F = kx$ with $F = 5$ and $x = 1$, so $k = 5$.

Thus the equation reads $10x'' + 5x = 0$ or, equivalently, $x'' + \frac{1}{2}x = 0$, from which the general solution is

$$x(t) = A\sin\frac{t}{\sqrt{2}} + B\cos\frac{t}{\sqrt{2}}.$$

Hence the resonance occurs at $\omega = 1/\sqrt{2}$ and $\omega = -1/\sqrt{2}$.

Answer: The resonance occurs at $\omega = 1/\sqrt{2}$, and, if we assume that ω can be negative, at $\omega = -1/\sqrt{2}$.

Problem 2.

a) Denoting $x_1 = x$, $x_2 = y$, $x_3 = x_1' = x'$, $x_4 = x_2' = y'$, we get $x'' = x_3'$ and $y'' = x_4'$ and hence we get the following

Answer:

$$\begin{aligned} x_1' &= x_3 \\ x_2' &= x_4 \\ x_3' &= 2x_3 - x_2 \\ x_4' &= 2x_4 - x_1. \end{aligned}$$

b) Writing the system in the matrix form, we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Next, we find the eigenvalues of the matrix. The sum of the eigenvalues is $0 + 0 = 0$ and the product is $0 \cdot 0 - (-3)(2) = 6$, so the eigenvalues are $i\sqrt{6}$ and $-i\sqrt{6}$. Let us find the corresponding eigenvectors.

For the eigenvalue $i\sqrt{6}$ the eigenvectors are found from the equation

$$\begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = i\sqrt{6} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{that is,} \quad \begin{aligned} 2y &= i\sqrt{6}x \\ -3x &= i\sqrt{6}y \end{aligned}$$

Hence we can choose for an eigenvector

$$\begin{bmatrix} 2 \\ i\sqrt{6} \end{bmatrix}$$

with the corresponding basic solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{i\sqrt{6}t} \begin{bmatrix} 2 \\ i\sqrt{6} \end{bmatrix} = \left(\cos(t\sqrt{6}) + i\sin(t\sqrt{6})\right) \begin{bmatrix} 2 \\ i\sqrt{6} \end{bmatrix}.$$

Taking the real imaginary parts, we get two basic solutions:

$$\begin{bmatrix} 2\cos(t\sqrt{6}) \\ -\sqrt{6}\sin(t\sqrt{6}) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2\sin(t\sqrt{6}) \\ \sqrt{6}\cos(t\sqrt{6}) \end{bmatrix}.$$

Thus the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 2\cos(t\sqrt{6}) \\ -\sqrt{6}\sin(t\sqrt{6}) \end{bmatrix} + C_2 \begin{bmatrix} 2\sin(t\sqrt{6}) \\ \sqrt{6}\cos(t\sqrt{6}) \end{bmatrix}$$

for some constants C_1 and C_2. Substituting the initial conditions $x(0) = 1$ and $y(0) = 2$ we get $C_1 = 1/2$ and $C_2 = 2/\sqrt{6}$, so we get the following

Answer: The solution is

$$x(t) = \cos\left(t\sqrt{6}\right) + \frac{4}{\sqrt{6}}\sin\left(t\sqrt{6}\right)$$
$$y(t) = -\frac{\sqrt{6}}{2}\sin\left(t\sqrt{6}\right) + 2\cos\left(t\sqrt{6}\right).$$

c) From the second equation, we get $2x = y + y'$, so $x = (y + y')/2$. Substituting the result in the first equation, we get

$$(y' + y'')/2 = -y - y' + 3y, \quad \text{that is,} \quad y'' + y' = 4y - 2y' \quad \text{and} \quad y'' + 3y' - 4y = 0.$$

We are looking for the solution in the form $y = e^{rt}$ for some r, which gives us the equation $r^2 + 3r - 4 = 0$ with the solutions $r = 1$ and $r = -4$. Hence

$$y = C_1 e^t + C_2 e^{-4t} \quad \text{for some} \quad C_1 \quad \text{and} \quad C_2.$$

Using that $x = (y + y')/2$, we get

$$x = C_1 e^t - \frac{3}{2} C_2 e^{-4t}.$$

Answer: The general solution is

$$\begin{aligned} x(t) &= C_1 e^t - 3C_2 e^{-4t} \\ y(t) &= C_1 e^t + 2C_2 e^{-4t} \end{aligned} \quad \text{where} \quad C_1 \quad \text{and} \quad C_2 \quad \text{can be arbitrary numbers.}$$

d) From the equations, we get

$$x'(0) = 2x(0) - y(0) = 2 - 2 = 0$$
$$y'(0) = 3x(0) + y(0) = 3 + 2 = 5,$$

so we approximate

$$x(1) = x(0) + x'(0) = 1$$
$$y(1) = y(0) + y'(0) = 7.$$

Now, from the equations,

$$x'(1) = 2x(1) - y(1) = 2 - 7 = -5$$
$$y'(0) = 3x(1) + y(1) = 3 + 7 = 10,$$

so we approximate

$$x(2) = x(1) + x'(1) = -4$$
$$y(2) = y(1) + y'(1) = 17.$$

Answer: The approximate values are $x(2) \approx -4$ and $y(2) \approx 17$.

Problem 3.
a) We have

$$\begin{aligned}AB &= \begin{bmatrix}1+i & 0\\ 2 & i\end{bmatrix}\begin{bmatrix}i & 2\\ 0 & 3\end{bmatrix} = \begin{bmatrix}(1+i)i+0\cdot 0 & (1+i)2+0\cdot 3\\ 2i+i\cdot 0 & 2\cdot 2+i\cdot 3\end{bmatrix}\\ &= \begin{bmatrix}-1+i & 2+2i\\ 2i & 4+3i\end{bmatrix} \quad \text{and}\\ BA &= \begin{bmatrix}i & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}1+i & 0\\ 2 & i\end{bmatrix} = \begin{bmatrix}i(1+i)+2\cdot 2 & i\cdot 0+2i\\ 0\cdot(1+i)+3\cdot 2 & 0\cdot 0+3i\end{bmatrix}\\ &= \begin{bmatrix}3+i & 2i\\ 6 & 3i\end{bmatrix}.\end{aligned}$$

Answer:

$$AB = \begin{bmatrix}-1+i & 2+2i\\ 2i & 4+3i\end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix}3+i & 2i\\ 6 & 3i\end{bmatrix}.$$

b) We have

$$\det\begin{bmatrix}3+i & -i\\ 6 & 2\end{bmatrix} = (3+i)\cdot 2-(-i)\cdot 6 = 6+2i+6i = 6+8i.$$

Answer: The determinant is equal to $6+8i$.

c) First, we find the eigenvalues. The sum of the eigenvalues is $-1+5=4$ and the product is $-1\cdot 5-(-9)\cdot 2=13$, from which one can guess the eigenvalues $2\pm 3i$. An equivalent way to find the eigenvalues is via the characteristic equation

$$\det\begin{bmatrix}-1-\lambda & 2\\ -9 & 5-\lambda\end{bmatrix} = 0, \quad \text{that is,} \quad \lambda^2-4\lambda+13=0,$$

from which we get again $\lambda = 2\pm 3i$. We find eigenvectors for eigenvalue $2+3i$ from the equation

$$\begin{bmatrix}-1 & 2\\ -9 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = (2+3i)\begin{bmatrix}x\\ y\end{bmatrix}, \quad \text{that is,} \quad \begin{aligned}-x+2y &= (2+3i)x\\ -9x+5y &= (2+3i)y\end{aligned} \quad \text{and hence}$$

$$\begin{aligned}(3+3i)x &= 2y\\ 9x &= (3-3i)y.\end{aligned}$$

The last two equations are equivalent: we get the second equation if we multiply the first one by $(3-3i)/2$. We get one eigenvector if, say, let $x=2$ and $y=(3+3i)$, other eigenvectors are non-zero multiples of

$$\begin{bmatrix}2\\ 3+3i\end{bmatrix}.$$

Since the matrix is real, to get the eigenvectors for eigenvalue $2-3i$ we take the complex conjugates of the eigenvectors for $2+3i$, which gives us the non-zero multiples of

$$\begin{bmatrix} 2 \\ 3-3i \end{bmatrix}.$$

Answer: The eigenvalues are $2+3i$ with the eigenvectors $\alpha \begin{bmatrix} 2 \\ 3+3i \end{bmatrix}$, where α can be any non-zero complex number and $2-3i$ with the eigenvectors $\beta \begin{bmatrix} 2 \\ 3-3i \end{bmatrix}$, where β can be any non-zero complex number.

d) First, we find the eigenvalues of the matrix

$$\begin{bmatrix} -1 & 13 \\ -2 & 1 \end{bmatrix}.$$

The sum of the eigenvalues is $-1+1=0$ and the product is $(-1)\cdot 1-(-2)\cdot 13=25$, so the eigenvalues are $\pm 5i$. Next, we find an eigenvector for eigenvalue $5i$, say. We set up the equation

$$\begin{bmatrix} -1 & 13 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5i \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{that is,} \quad \begin{aligned} (1+5i)x &= 13y \\ 2x &= (1-5i)y. \end{aligned}$$

The last two equations are equivalent: we obtain the second equation by multiplying the first equation onto $(1-5i)/13$. Setting, for example, $y=2$ and $x=(1-5i)$ we obtain an eigenvector and a basic complex solution

$$e^{5it} \begin{bmatrix} 1-5i \\ 2 \end{bmatrix} = (\cos 5t + i \sin 5t) \begin{bmatrix} 1-5i \\ 2 \end{bmatrix}.$$

To get basic real solutions, we take the real part

$$\begin{bmatrix} \cos 5t + 5\sin 5t \\ 2\cos 5t \end{bmatrix}$$

and the imaginary part

$$\begin{bmatrix} -5\cos 5t + \sin 5t \\ 2\sin 5t \end{bmatrix}$$

of the basic complex solution. Hence the general solution is

$$\begin{bmatrix} (t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} \cos 5t + 5\sin 5t \\ 2\cos 5t \end{bmatrix} + C_2 \begin{bmatrix} -5\cos 5t + \sin 5t \\ 2\sin 5t \end{bmatrix},$$

where C_1 and C_2 can be arbitrary constants.

Answer: The general solutions are

$$\begin{aligned} x(t) &= (C_1 - 5C_2)\cos 5t + (5C_1 + C_2)\sin 5t \\ y(t) &= 2C_1 \cos 5t + 2C_2 \sin 5t, \end{aligned}$$

where C_1 and C_2 can be arbitrary real numbers.

Problem 4.

a) First, we find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}.$$

The sum of the eigenvalues is $-3 + (-2) = -5$ and the product of the eigenvalues is $(-3) \cdot (-2) - 1 \cdot 2 = 4$, from which the eigenvalues are -1 and -4. Next, we find an eigenvector for eigenvalue -1 from the system

$$\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} x \\ y \end{bmatrix},$$

and an eigenvector for eigenvalue -4 from the system

$$\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -4 \begin{bmatrix} x \\ y \end{bmatrix}.$$

For example, we can pick up $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as an eigenvector for eigenvalue -1 and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as an eigenvector for eigenvalue -4. Since $\sqrt{-1} = \pm i$ and $\sqrt{-4} = \pm 2i$, we get two basic complex solutions

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{it} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and}$$
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{2it} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Taking the real and imaginary parts, we get four basic real solutions

$$\begin{bmatrix} \cos t \\ 2\cos t \end{bmatrix}, \begin{bmatrix} \sin t \\ 2\sin t \end{bmatrix}, \begin{bmatrix} \cos 2t \\ -\cos 2t \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \sin 2t \\ -\sin 2t \end{bmatrix}.$$

The general solution is a combination of the four basic solutions with arbitrary real coefficients, hence we get the following

Answer: The general solution is

$$x(t) = C_1 \cos t + C_2 \sin t + C_3 \cos 2t + C_4 \sin 2t$$
$$y(t) = 2C_1 \cos t + 2C_2 \sin t - C_3 \cos 2t - C_4 \sin 2t,$$

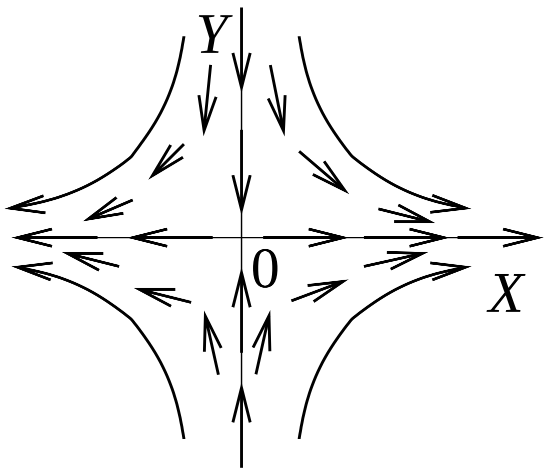
where C_1, C_2, C_3 and C_4 are arbitrary real numbers.

b) We get the equation

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = -\frac{2y}{3x}$$

for the slope field.

Hence for $x = 0$ (the y-axis) the slope is infinite (the direction field is represented by vertical arrows) and for $y = 0$ (the x-axis) the slope is 0 (the direction field is represented by a horizontal arrows). Since $dy/dt = -2y$ we deduce that on the y-axis the direction field is represented by vertical arrows pointing towards the origin: y decreases for positive and increases for negative y. Since $dx/dt = 3x$ we deduce that on the x-axis the direction field is represented by horizontal arrows pointing from the origin: x decreases for negative and increases for positive x. We then interpolate the picture onto the whole plane and get something like this:



Arrows show the direction field and solid curves are trajectories.

c) We compute the eigenvalues of the matrix

$$\begin{bmatrix} 0 & -5 \\ -2 & 0 \end{bmatrix}.$$

The sum of the eigenvalues is 0 and the product is $0-(-2)\cdot(-5) = -10$, so the eigenvalues are $\sqrt{10}$ and $-\sqrt{10}$. Since one of the eigenvalues is positive, the critical point $(0, 0)$ is unstable.

Answer: The critical point $(0, 0)$ is unstable.

d) We find the critical points by setting up the system of equations

$$\begin{aligned} x + x^2 - 2xy &= 0 \\ y + 2y^2 - 3xy &= 0 \end{aligned} \quad \text{that is,} \quad \begin{aligned} x(1 + x - 2y) &= 0 \\ y(1 + 2y - 3x) &= 0. \end{aligned}$$

This gives us the following possibilities:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{or}$$

$$\begin{cases} x = 0 \\ 1 + 2y - 3x = 0 \end{cases} \Longrightarrow \begin{cases} x = 0 \\ y = -1/2 \end{cases} \qquad \text{or}$$

$$\begin{cases} 1 + x - 2y = 0 \\ y = 0 \end{cases} \Longrightarrow \begin{cases} x = -1 \\ y = 0 \end{cases} \qquad \text{or}$$

$$\begin{cases} 1 + x - 2y = 0 \\ 1 + 2y - 3x = 0 \end{cases} \Longrightarrow \begin{cases} 2 - 2x = 0 \quad \text{(we add the two equations)} \\ 1 + 2y - 3x = 0 \end{cases}$$
$$\Longrightarrow \begin{cases} x = 1 \\ y = 1. \end{cases}$$

Answer: The critical points are $(0, 0)$, $(0, -1/2)$, $(-1, 0)$, and $(1, 1)$.