SOLUTIONS TO MATH 216 FIRST MIDTERM EXAM

1. Problem. We separate the variables:

$$\frac{dy}{dx} = xy^2 \Longrightarrow \frac{dy}{y^2} = x \quad \text{(check whether } y = 0 \text{ is a solution)}$$
$$\Longrightarrow \int \frac{dy}{y^2} = \int x\, dx \Longrightarrow -\frac{1}{y} = \frac{x^2}{2} + C$$
$$\Longrightarrow y = -\frac{1}{\frac{x^2}{2} + C}.$$

As we divided by y^2, we must check whether $y = 0$ is a solution and indeed it is.

Answer: The solutions are

$$y = -\frac{2}{C + x^2},$$

where C can be any number, and

$$y = 0.$$

2. Problem. One way to match the equations with their slope fields is to look where the slopes are 0 (the directions are horizontal). Hence we get the following

Answer:

1. $y' = x^2$, Figure F (the slope is 0 whenever $x = 0$),
2. $y' = y^2$, Figure C (the slope is 0 whenever $y = 0$),
3. $y' = y - x$, Figure E (the slope is 0 whenever $y = x$),
4. $y' = x + y$, Figure D (the slope is 0 whenever $y = -x$),
5. $y' = x^2 + y^2$, Figure A (the slope is 0 only if $x = y = 0$),
6. $y' = x^2 - y^2$, Figure B (the slope is 0 if $y = x$ or if $y = -x$).

3. Problem.

a) Since the tank gets $5 - 4 = 1$ gallon each second, it will be full in 100 seconds.

Answer: The tank is full in 100 seconds.

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b) In t seconds, the tank contains $100+t$ gallons of brine. Therefore, in t seconds, the gallon of brine contains $x(t)/(100+t)$ pounds of salt. Therefore, we get the equation

$$\frac{dx}{dt} = 5 - 4\frac{x}{100+t}$$

with the initial condition $x(0) = 50$.

Answer: The equation is

$$\frac{dx}{dt} = 5 - 4\frac{x}{100+t}.$$

c) We have a linear first-order equation

$$x' + \frac{4}{100+t}x = 5.$$

The integrating factor is

$$\rho(t) = \exp\left(\int \frac{4}{100+t}\,dt\right) = \exp\left(4\ln(100+t)\right) = (100+t)^4.$$

Multiplying by $\rho(t)$, we get

$$\begin{aligned}
&(100+t)^4x' + 4(100+t)^3x = 5(100+t)^4\\
\Longrightarrow &\frac{d}{dt}\left((100+t)^4x(t)\right) = 5(100+t)^4\\
\Longrightarrow &(100+t)^4x(t) = \int 5(100+t)^4\,dt = (100+t)^5 + C\\
\Longrightarrow &x(t) = (100+t) + \frac{C}{(100+t)^4}.
\end{aligned}$$

Since $x(0) = 50$ we get

$$50 = 100 + \frac{C}{100^4} \quad \text{and} \quad C = (-50)\cdot 100^4.$$

Therefore,

$$x(t) = (100+t) - \frac{50\cdot 100^4}{(100+t)^4}.$$

The tank is full in 100 seconds. At that time

$$x(100) = 200 - \frac{50\cdot 100^4}{200^4} = 200 - \frac{50}{16} = 196\frac{7}{8} = 196.875.$$

Answer: When the tank is full it contains $196\frac{7}{8} = 196.875$ lbs of salt.

4. Problem. One can solve this equation by separation of variables or as a first-order linear equation. Let us solve it as a first-order linear equation.

The integrating factor is

$$\rho(x) = \exp\left(\int x\ dx\right) = e^{x^2/2}.$$

Hence we get

$$\begin{aligned} & y' + xy = x \Longrightarrow e^{x^2/2}y' + xe^{x^2/2}y = xe^{x^2/2} \\ \Longrightarrow & \left(e^{x^2/2}y\right)' = xe^{x^2/2} \Longrightarrow e^{x^2/2}y = \int xe^{x^2/2}\ dx = e^{x^2/2} + C \\ \Longrightarrow & y = 1 + Ce^{-x^2/2}. \end{aligned}$$

Since $y(0) = 2$ we get $C = 1$.

Answer: $y(x) = 1 + e^{-x^2/2}$.

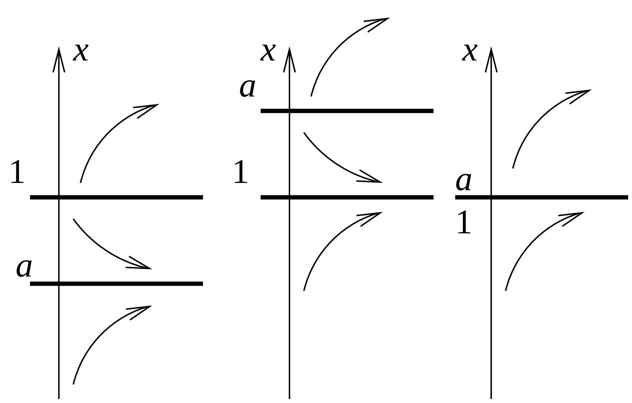
5. Problem.

a) An equilibrium solution is a solution of the type $x(t) = c$, where c is a number. Thus we must have $(c - a)(c - 1) = 0$, so $c = a$ or $c = 1$.

Answer: The equilibria solutions are $x(t) = 1$ and $x(t) = a$.

b) Suppose that $a < 1$. If $x < a$ then $x < 1$ and hence $x' = (x - a)(x - 1) > 0$, so x is increasing and approaching a. If $1 > x > a$ then $x' = (x - a)(x - 1) < 0$ so x is decreasing and approaching a. Hence $x(t) = a$ is stable.

If $x > 1$ then $x > a$ and hence $x' = (x - a)(x - 1) > 0$. Therefore, x is increasing. Therefore, $x(t) = 1$ is unstable.



Suppose that $a > 1$. If $x < 1$ then $x < a$ and hence $x' = (x - a)(x - 1) > 0$. Therefore x is increasing and approaching 1. If $a > x > 1$ then $x' = (x - a)(x - 1) < 0$. Therefore x is decreasing and approaching 1. Hence $x(t) = 1$ is stable.

If $x > a > 1$ then $x' = (x - a)(x - 1) > 0$. Therefore x is increasing. Hence $x(t) = a$ is unstable.

Suppose that $a = 1$. If $x \neq 1$ then $x' = (x - a)(x - 1) = (x - 1)^2 > 0$ and hence is x is increasing. Therefore, $x(t) = a = 1$ is semi-stable.

Answer: If $a < 1$ then $x(t) = a$ is a stable equilibrium and $x(t) = 1$ is an unstable equilibrium. If $a > 1$ then $x(t) = a$ is an unstable equilibrium and $x(t) = 1$ is a stable equilibrium. If $a = 1$ then $x(t) = 1 = a$ is a semi-stable equilibrium.

6. Problem. Let $v(t)$ be the speed of the boat at time t. Then we have

$$\frac{dv}{dt} = 25 - kv^2,$$

where k is some positive number. Besides, $v(0) = 0$. As long as $25 - kv^2 > 0$ the speed is increasing. Hence for the limiting speed v we must have $25 = kv^2$. Therefore, $25 = k \cdot 100$, so $k = 1/4$ and the equation is

$$\frac{dv}{dt} = 25 - \frac{1}{4}v^2.$$

We separate the variables:

$$\frac{dv}{100 - v^2} = \frac{1}{4}dt \Longrightarrow \int \frac{dv}{100 - v^2} = \frac{1}{4}\int dt = \frac{1}{4}t + C.$$

Using partial fractions, we compute

$$\int \frac{dv}{100 - v^2} = \int \frac{dv}{(10 - v)(10 + v)} = \int \frac{1}{20}\left(\frac{1}{10 - v} + \frac{1}{10 + v}\right) dv$$
$$= \frac{1}{20} \ln \left| \frac{10 + v}{10 - v} \right|.$$

As we are interested in the velocity between 0 and 10, we can drop the absolute value sign and hence we get

$$\ln \frac{10 + v}{10 - v} = 5t + C.$$

Substituting $v(0) = 0$, we get $C = 0$, so finally

$$\ln \frac{10 + v}{10 - v} = 5t \Longrightarrow \frac{10 + v}{10 - v} = e^{5t} \Longrightarrow (10 + v) = e^{5t}(10 - v)$$
$$\Longrightarrow \left(1 + e^{5t}\right) v = 10\left(e^{5t} - 1\right) \Longrightarrow v(t) = 10\left(\frac{e^{5t} - 1}{e^{5t} + 1}\right).$$

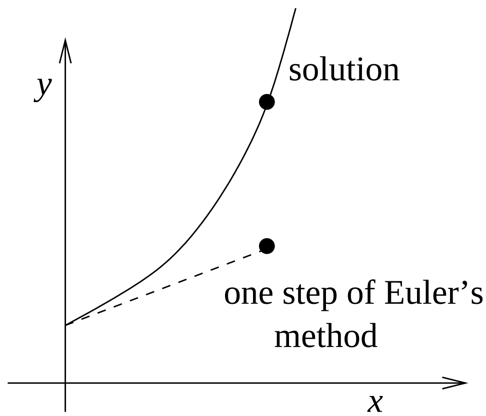
We are interested at the time t when $v(t) = 6$. Hence we get the equation

$$10\left(\frac{e^{5t} - 1}{e^{5t} + 1}\right) = 6 \Longrightarrow \frac{e^{5t} - 1}{e^{5t} + 1} = \frac{3}{5} \Longrightarrow e^{5t} - 1 = \frac{3}{5}\left(e^{5t} + 1\right) \Longrightarrow e^{5t} = 4$$
$$\Longrightarrow t = \frac{1}{5}\ln 4.$$

Answer: The boat reaches the speed of 6 ft/sec in $(\ln 4)/5 \approx 0.28$ sec.

7. Problem.

a) If $x \geq 0$ and $y \neq 0$ then $y' > 0$ so $y(x)$ is increasing. Since $y(x)$ is increasing, $y'(x)$ is increasing as well and $y(x)$ is concave up. Euler's method uses the slope at the left end of the interval as an estimate for the average slope on the interval. Since the solution $y(x)$ is concave up, the slope elsewhere on the interval will be bigger than the slope at the left end of the interval and Euler's method will produce an underestimate.



Answer: Euler's method will underestimate the value of $y(0.2)$.

b) We estimate

$$
\begin{aligned}
y'(0) = 0 + y^2(0) = 0 + 1^2 = 1, \quad \text{so} \quad y(0.1) &= y(0) + 0.1 \cdot y'(0) \\
&= 1 + 0.1 \cdot 1 = 1.1; \\
y'(0.1) = 0.1 + y^2(0.1) = 0.1 + (1.1)^2 = 1.31, \quad \text{so} \quad y(0.2) &= y(0.1) + 0.1 \cdot y'(0.1) \\
&= 1.1 + 0.1 \cdot 1.31 = 1.231.
\end{aligned}
$$

Answer: Euler's method estimates $y(0.2) = 1.231$.