Coloring Knots

This project is more algebra than topology. Knot theory can be studied combinatorially, and you don’t need to know the topological details if you accept the combinatorial outcome. Roughly, then:

A “knot” is a smooth embedding of a circle into \(\mathbb{R}^3\). Two knots are “equivalent” if one can be deformed to the other smoothly. The “knot class” of a given knot is the set of knots equivalent to it.

A “knot projection” is a map from a circle to the plane which is smooth except for finitely many points at which the image crosses itself transversally, and at each of these points an “undercrossing” is chosen. A generic orthogonal projection of a knot onto a plane yields a knot projection.

As with the knots themselves, two knot projections are “equivalent” if one can be deformed into another smoothly, but equivalent knots have many inequivalent projections. The passage from one equivalence class of projections of a knot class to any another can be made by a sequence of “Reidemeister moves”.

There are three types of Reidemeister moves:

\[
\begin{align*}
&\text{I} & \quad \text{II} & \quad \text{III} \\
&\text{\includegraphics[width=0.2\textwidth]{reidemeister1.png}} & \quad \text{\includegraphics[width=0.2\textwidth]{reidemeister2.png}} & \quad \text{\includegraphics[width=0.2\textwidth]{reidemeister3.png}}
\end{align*}
\]

The maximal paths not containing any undercrossings are the “arcs” of the knot projection. A “3-coloration” is a function from the set of arcs of a knot projection to the set \{red, green, blue\}, with the property that, at each crossing, the three arcs meeting there have the same color, or else have all different colors.

A 3-coloration is “nontrivial” if it involves more than just one color.

Show that admitting a nontrivial 3-coloration is a knot invariant.

Give some examples: prove that some knots are not equivalent to each other by this method.

The “sum” of two knots is obtained by breaking each and tying the two together. Questions: how well-defined is this sum? How does colorability relate to this sum?

A knot is “prime” if it cannot be written as sum of two nontrivial knots. A table of prime knots is attached.

The basic project is to explore the following question: How does the notion of 3-colorability generalize?