Continued Fractions

This project is not suitable for students who have learned about continued fractions before.

The continued fraction expansion of a real number \(x > 1\) looks like this:

\[
a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}
\]

where the \(a_i\) are positive integers that are obtained recursively. Starting with \(x_0 = x\), the recursive definitions are

\[a_n = \lfloor x_n \rfloor \quad \text{and} \quad x_{n+1} = \frac{1}{x_n - a_n},\]

where the notation \(\lfloor x \rfloor\) stands for the largest integer \(\leq x\).

The **convergents** are the fractions

\[
a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \frac{1}{a_{n-1}}}}}
\]

We write them in the form \(\frac{p_n}{q_n}\), where \(p_n\) and \(q_n\) are relatively prime positive integers.

**Assignment**

1. Compute continued fraction expansions of some rational numbers, and some other numbers such as \(\sqrt{2}\), \(\sqrt{5}\), \(\sqrt{7}\), \(\pi\), \(e\), \(\frac{e^2 - 1}{e + 1}\). Find some patterns and prove some theorems!

2. Do the “convergents” in fact converge to something? If so, can you make precise how quickly they converge?

3. Can the values \(p_n\) and \(q_n\) be computed efficiently from the \(a_i\)’s?