Periodic Recurrence Relations

We can define a sequence \( a_0, a_1, a_2, \ldots \) recursively using a function \( F(x, y) \) of two variables. To do this, we assign initial values \( a_0, a_1 \) and then use the recurrence relation

\[
a_{n+1} = F(a_n, a_{n-1}) \tag{1}
\]

For this definition to make sense, the function should be defined almost everywhere.

The recursive definition is periodic if, with indeterminate initial values \( a_0 = x \) and \( a_1 = y \), there is an integer \( n > 0 \) so such that \( a_n = x \) and \( a_{n+1} = y \) identically. The period of the sequence is the smallest such integer \( n \).

We can compute easily by hand when \( F \) is the function \( F(x, y) = x^{-1} \):

\[
a_0 = x, \quad a_1 = y, \quad a_2 = x^{-1}, \quad a_3 = y^{-1}, \quad a_4 = x, \quad a_5 = y \tag{2}
\]

The sequence is periodic, of period 4.

This project looks at the behavior of the sequence \( a_0, a_1, a_2, \ldots \) for different functions \( F \). The two main questions are:

- Which functions \( F \) define periodic sequences?
- Which functions \( F \) define bounded sequences?

Begin your investigation by studying the following three special cases

(I) Let \( F \) be a function given by the formula

\[
F(x, y) = \frac{f(y)}{x} \tag{3}
\]

where \( f \) is a polynomial or a rational function in one variable. For such a function, the recurrence (1) becomes

\[
a_{n+1} = \frac{f(a_n)}{a_{n-1}}
\]

(II) Let \( F \) be a function given by the formula

\[
F(x, y) = |y| - x
\]

or more generally

\[
F(x, y) = c|y| - x \tag{4}
\]

where \( c > 0 \) is a constant. For such a function the recurrence becomes

\[
a_{n+1} = c|a_n| - a_{n-1}.
\]
Suppose that our sequence is defined by two different functions, namely

$$a_{n+1} = \begin{cases} 
\frac{a_n^b + 1}{a_{n-1}} & \text{if } n \text{ is odd} \\
\frac{a_n^c + 1}{a_{n-1}} & \text{if } n \text{ is even}
\end{cases}$$

(5)

where $b$ and $c$ are constants.

Some sample questions to get you started:

1. Are there rational functions $f$ for which the recurrence relation (3) has period 2?
2. For what polynomials $f$ does the recurrence relation (3) have period 4?
3. What can be said about other periods of the recurrence relation (3)?
4. Are there types of functions $f$ which can be ruled out a priori – some property shows that they can’t produce a periodic sequence?
5. For recurrence relations (4) and (5), which values of parameters yield periodic sequences?
6. For the general recurrence relation (1), what can be said about polynomial functions $F(x, y)$ that yield periodic sequences?
7. If the recurrence is not periodic, what can one say about boundedness of the solutions. Find conditions under which they always remain bounded.