Polynomial Images of Circles

The Fundamental Theorem of Algebra states that every nonconstant polynomial \( f(z) = a_n z^n + \cdots + a_0 \) has a root in the complex numbers. One proof of this theorem considers the map \( w = f(z) \) from the complex \( z \)-plane to the complex \( w \)-plane. Let \( C_r \) denote the circle of radius \( r \) about the origin in the \( z \)-plane. Whenever \( C_r \) passes through a root of the polynomial, its image \( f(C_r) \) passes through the origin.

Project, Part I

1. Set up a graphing program so that, given a polynomial \( f(z) \), you can graph the image \( f(C_r) \), allowing \( r \) to vary.

2. Using the polynomial \( f(z) = z^3 + iz^2 + (1 + i)z - (2 + 2i) \) to start with, watch the image curve \( f(C_r) \), as \( r \) shrinks from a large value to a very small one. Experiment with other polynomials. Several interesting things happen. Explain them.

Project, Part II

The total curvature of a smooth path \( D \) is \( \int_D \kappa \, ds \), the integral of the curvature \( \kappa \). The total curvature of a closed path is a multiple of \( 2\pi \). Explain why this is so. Make a program to compute the total curvature of \( f(C) \), when \( f \) is a polynomial and \( C \) is an arbitrary closed path, and try to find the geometric meaning of the total curvature of \( f(C) \).