Random Walks

Consider a random walk on the integer lattice in the plane. If a “particle” making a random walk arrives at a lattice point \( p = (k_1, k_2) \) at the time \( t \), then one of the four neighbors \((k_1 \pm 1, k_2), (k_1, k_2 \pm 1)\) of \( p \) is selected with equal probability \( \frac{1}{4} \). The particle moves to that neighbor at time \( t + 1 \).

Let \( D \) be a region in the plane (a square or a half plane for example), and let \( B \) denote its boundary. Let \( p \) be a point of \( D \), and let \( b \) be a boundary point. We’ll denote by \( P_p(b) \) the probability that a random walk starting at \( p \) exits at \( b \), i.e., that \( b \) is the first boundary point that is reached.

Many questions can be asked.

- What is the probability that a particle starting at \( p \) never reaches the boundary?
- What is the “exit time”, the expected time for a particle starting at \( p \) to reach the boundary?
- How does the exit time depend on \( p \)?
- Holding \( p \) fixed, what is the function \( P_p(b) \)?
- If \( D \) is a square, what is the behavior of \( P_p(b) \) near the corners?
- Holding \( b \) fixed, \( P_p(b) \) becomes a function of \( p \). What are the properties of this function?

Finally, suppose that \( D \) is the half plane \( y \geq 0 \). Consider the lattice spanned by the vectors \( \frac{1}{2} e_1, \frac{1}{2} e_2 \). One obtains a continuous density function \( f \) on the \( x \)-axis by passing to a limit. At a lattice point \( b = \frac{a}{2n} \),

\[
  f(b) = \lim_{k \to \infty} 2^k P_p(b).
\]

- What is this density function?

Investigate as many of these questions as you can, when \( D \) is the square \( 0 \leq x, y \leq n \), and the half plane \( 0 \leq y \). If time permits, make some other experiments.