Math 412 - Problem Set 6
Many of these questions are taken from the book or from Hungerford, “Abstract Algebra.”

1) Determine whether the following subsets are subrings of the given rings $R$ and, if so, whether they are ideals.
   (a) The subset of $R = \mathbb{Z}[x]$ consisting of constant functions (i.e. polynomials with no terms of degree greater than 0).
   (b) The subset \{0, 2, 3, 6\} in $R = \mathbb{Z}_{12}$.
   (c) The subset of the ring $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ consisting of numbers of the form $2a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$.
   (d) The subset $A = \{a + 2bi \mid a, b \in \mathbb{Z}\}$ of the ring $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$.
   (e) For any ring $S$, the subset of $R = S \oplus S$ given by $I = \{(s, s) \mid s \in S\}$.
   (f) For any commutative ring $R$ with identity and any element $a \in R$, the subset $I = \{ra \mid r \in R\}$.

2) Show that the only ideals of $M_2(\mathbb{R})$ are the entire ring and the zero ideal.

3) (a) Show that the subset $I$ of $\mathbb{Z}[x]$ consisting of polynomials with zero constant term is an ideal.
   (b) Show that the quotient ring $\mathbb{Z}[x]/I$ is isomorphic to $\mathbb{Z}$.

4) (a) Prove the the set $R$ of upper-triangular matrices (i.e. those matrices with a 0 in the lower left corner) is a subring of $M_2(\mathbb{R})$.
   (b) Prove that the set $I$ of matrices of the form $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ is an ideal of $R$ (not of $M_2(\mathbb{R})$!).
   (c) Describe the cosets of $I$. The quotient ring $R/I$ should look familiar - what ring is it isomorphic to? Prove that they are isomorphic.

5) Prove or disprove: If $R$ is a ring with subrings $J$ and $I$ such that $I$ is an ideal of $J$, then $I$ is an ideal of $R$.

6) Let $R$ be a ring and let $I_1, I_2 \subset R$ be ideals of $R$. Let $I = I_1 \cap I_2$.
   (a) Prove that $I$ is an ideal of $R$.
   (b) Prove that $I$ is an ideal of $I_1$ and of $I_2$.

7) Let $R$ be a commutative ring with identity. An ideal $I$ of $R$ is called prime if whenever $a$ and $b$ are elements of $R$ with $ab \in I$ then either $a \in I$ or $b \in I$.
   (a) Show that for a positive integer $n$ the ideal $n\mathbb{Z} \subset \mathbb{Z}$ is prime if and only if $n$ is prime.
   (b) Show that $I$ is a prime ideal if and only if $R/I$ is an integral domain.

8) Show that every non-zero homomorphism $f : \mathbb{Z} \to M_2(\mathbb{R})$ is injective. (Hint: use the First Isomorphism Theorem.)