1) Perform the following operations and simplify your answer.


(b) In \( \mathbb{Q}[x] \), divide \(2x^4 - x^3 + 6x^2 + 3\) by \(x^2 - 1\).

(c) In \( \mathbb{R}[x] \), divide \(x^5 + 2x^4 + x + 2\) by \(x + 2\).


2) Let \( F \) be a field.

(a) What are the units in the ring \( F[x] \)?

(b) Choose a polynomial \( f(x) \in F[x] \) and a unit \( g(x) \) in \( F[x] \). If we apply the division algorithm to divide \( f(x) \) by \( g(x) \), what is the result?

3) Let \( R \) be a ring and let \( I \) be an ideal. Show that \( R/I \) is commutative if and only if for every \( a, b \in R \) we have \( ab - ba \in I \).

4) Let \( R \) be a ring and let \( I \) be an ideal.

(a) Prove that if \( J \subset R \) is an ideal containing \( I \) then the image of \( J \) under the canonical map \( f : R \to R/I \) is an ideal.

(b) Prove that the preimage (under the canonical map as in (a)) of an ideal in the ring \( R/I \) is an ideal \( J \subset R \) containing \( I \).

Since the operations in (a) and (b) are inverses, this gives a bijection between ideals of \( R \) containing \( I \) and ideals of \( R/I \).

(c) Explicitly demonstrate this bijection when \( R = \mathbb{Z} \) and \( I = (24) = 24\mathbb{Z} \).

5) Consider the ideal \( I \) generated by the polynomial \( x^2 - 2 \) in \( \mathbb{Q}[x] \). Show that the quotient ring \( \mathbb{Q}[x]/I \) is isomorphic to the field \( R = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\} \).

6) Define the function \( D : \mathbb{R}[x] \to \mathbb{R}[x] \) to be the derivative map:

\[
D(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) = a_1 + 2a_2 x + \ldots + na_n x^{n-1}.
\]

Is \( D \) a homomorphism? Is it an isomorphism? Discuss what the First Isomorphism Theorem tells you in this situation.