UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Algebra
8 January 2005: Morning Session, 9:00-12:00

Answer all questions. Prove your answers.

1. Let G be a group (possibly nonabelian, written multiplicatively) with subgroups H and K of finite, relatively prime indices. Prove that $G = HK$.

2. (a) Determine the elementary divisors d_1, d_2, d_3 of the matrix

$$A = \begin{pmatrix} -1 & 2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 5 \end{pmatrix}.$$

(b) Let x_1, x_2, x_3 be indeterminates and define $y_1 = x_1^{-1}x_2^2x_3$, $y_2 = x_1^{-1}x_3$ and $y_3 = x_1^{-1}x_2^{-2}x_3^5$. (Note that the exponents are the entries of the matrix A.) Prove that the field extension $\mathbb{C}(x_1, x_2, x_3) : \mathbb{C}(y_1, y_2, y_3)$ is a Galois extension with Galois group $\mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \mathbb{Z}/d_3\mathbb{Z}$.

(c) How many fields are there strictly in between $\mathbb{C}(x_1, x_2, x_3)$ and $\mathbb{C}(y_1, y_2, y_3)$? (You do not have to give these fields explicitly, but you do have to motivate your answer.)

3. A 5×5 mystery matrix A with complex entries has the following properties: $\det(A) = 1$, $\operatorname{trace}(A) = 5$, $\operatorname{rank}(A - I) = 3$ and the minimum polynomial of A has degree 3. What are the possibilities for the Jordan canonical form of A?

4. Let $q = p^k$ where p is a prime number and k is a positive integer. The group $\mathrm{GL}_n(\mathbb{F}_q)$ of invertible matrices over the field $\mathbb{F}_q$ with q elements has cardinality

$$(q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1}).$$

(a) Let $U_n(\mathbb{F}_q)$ be the subgroup of upper triangular matrices with 1's on the diagonal. Show that $U_n(\mathbb{F}_q)$ is a p-sylow subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$.

(b) Suppose that H is a p-group of order p^r. Show that for all q, H is isomorphic to a subgroup of $U_n(\mathbb{F}_q)$ with $n = p^r$.

(c) Suppose that $\mathrm{GL}_n(\mathbb{F}_q)$ contains the cyclic group of order p^r. Prove that $n > p^{r-1}$.

5. Let V_n be the $n \times n$ symmetric matrices with real entries. Let $D \in V_n$ be a fixed diagonal matrix. Define a bilinear form $B_D(\cdot, \cdot)$ on V_n by

$$B_D(X, Y) := \operatorname{trace}(XDY).$$

(a) Prove that B_D is symmetric.

(b) Determine conditions on the diagonal entries of D that characterize when B_D is non-degenerate.

(c) Determine the rank and signature of B_D if $n = 3$ and the diagonal entries of D are $0, 1$, and -1.

UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Algebra
8 January 2005: Afternoon Session, 2:00-5:00

Answer all questions. Prove your answers.

1. Consider these permutations in G, the symmetric group on $\{1,2,3,4,5,6,7,8,9\}$. Let H be the subgroup generated by $\alpha := (1,2,3)$ and $\beta := (1,6,2,7,3,8,9)$.

(a) Describe the orbits of H on $\{1,2,3,4,5,6,7,8,9\}$.
(b) Prove that $|H|$ divides $2^3 \cdot 3^2 \cdot 5 \cdot 7$.
(c) Prove that $|H|$ is divisible by $2^2 \cdot 3^2 \cdot 5 \cdot 7$.

2. Let F be a field of characteristic 0.

(a) Suppose that F contains a primitive n^{th} root of unity. Let $a \in F$ and let E be a splitting field for $x^n - a$ over F. Prove that the Galois group of E/F is an abelian group which is isomorphic to a subgroup of $\mathbb{Z}/n\mathbb{Z}$.

(b) Let F be a field of characteristic 0. Let $a \in F$ and let E be a splitting field for $x^n - a$ over F. Prove that the Galois group of E/F is a solvable group.

3. Let V be a finite dimensional vector space over an algebraically closed field F and $T \in \mathrm{End}_F(V)$.

(a) Prove that there is a well defined linear transformation $S \in \mathrm{End}_F(\wedge^2 V)$ so that $S(v \wedge w) = Tv \wedge Tw$.

(b) If the characteristic of F is not 2, find a formula for the trace of S in terms of $\mathrm{tr}(T)$ and $\mathrm{tr}(T^2)$, where tr means trace on V.

4. Let F be a field and p a prime number. Let M be the permutation matrix for a p-cycle,

$$\text{e.g., } M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

(a) Find the rational canonical form for M.

(b) Assume that the characteristic polynomial for M factorizes completely into linear factors in $F[x]$. Find the Jordan canonical form.

(c) Suppose that $F = \mathbb{F}_2$, the field of integers modulo 2 and S is an $n \times n$ matrix whose minimal polynomial is $m(x) = (x^2 + x + 1)(x^3 + x^2 + 1)$. For the action of S on $V := \mathbb{F}_2^n$, there is a decomposition of V into summands V_i corresponding to the irreducible factors of $m(x)$ in $F[x]$. Show how to compute the projections to the V_i, as particular polynomial expressions in S.

5. A group G is called nilpotent if there is a series $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ so that $G_i/G_{i-1} \leq Z(G/G_{i-1})$.

When G is finite, prove that this is equivalent to G being the direct product of its Sylow p-groups, for the primes p which divide $|G|$.