UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Algebra
6 May 2005: Morning Session, 9:00-12:00

(1) Let v be a nonzero integer $n \times 1$ column vector, and define the matrix $B := vv^T$. Determine the elementary divisors (Smith invariants) of B.

(2) Consider the polynomial

$$f(X) = X^4 + rX^2 + s \in \mathbb{Q}[X]$$

and let K be the splitting field of f over $\mathbb{Q}$.

(a) Show that the the degree of the extension $K : \mathbb{Q}$ is either 1, 2, 4 or 8.

(b) Show that $f(X)$ is irreducible if and only if $r^2 - 4s$ is not a square (of an element in $\mathbb{Q}$), and either s is not a square or s is a square and both $2\sqrt{s} - r$ and $-2\sqrt{s} - r$ are not squares.

(c) Compute $\text{Gal}(K : \mathbb{Q})$ if $r = s = -1$.

(3) Let G be a finite solvable group.

(a) Suppose that K is a minimal normal subgroup of G. Prove that K is an elementary abelian p-group, for some prime, p, i.e., a direct product of cyclic groups of order p;

(b) Suppose that M is a maximal subgroup of G. Prove that $|G : M|$ is a power of a prime.

(4) Let K be a field. Suppose that V is a finite dimensional K-vector and $A : V \to V$ is an endomorphism. A vector $v \in V$ is called cyclic (for A) if V is spanned by $v, Av, A^2v, \ldots$.

(a) Show that if there exists a cyclic vector for A, then the minimum polynomial and the characteristic polynomial of A are the same.

(b) Suppose there exists a cyclic vector and $B : V \to V$ is an endomorphism which commutes with A (i.e., $AB = BA$). Show that there exists a polynomial $p \in K[X]$ such that $B = p(A)$.

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(5) Let G be a nonabelian simple group and let G_1, G_2, G_3 be isomorphic copies of G.

(a) Prove that the complete set of normal subgroups of $G_1 \times G_2$ is

$$\{1, G_1, G_2, G_1 \times G_2\}.$$

(We identify G_1 with $\{(x, 1) \in G_1 \times G_2 | x \in G_1\}$, etc.)

(b) Now assume that G is finite. Let φ be a monomorphism of groups,

$$\varphi : G_1 \times G_2 \to G_1 \times G_2 \times G_3.$$

Prove that there exist indices i and j, not necessarily distinct, so that $\varphi(G_i) = G_j$.

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6 May 2005: Afternoon Session, 2:00-5:00

(1) Suppose that $K : \mathbb{Q}$ is a Galois extension. Prove that for any subfield L of K there exist subfields $L_1, L_2, \ldots, L_r \subseteq K$ such that $L = L_1 \cap \cdots \cap L_r$ and the degree of the field extension $K : L_i$ is a prime power for all i.

(2) Let H be a group of order $5^p \cdot 7^q$, for integers $p \geq 0, q \geq 0$.
(a) Show that an action of H on a set with 23 elements has a fixed point.
(b) Suppose that H, as above, is a subgroup of index 24 in the larger group G. Prove that the normalizer

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

is strictly larger than H.

(3) Given a pair of linear transformations $T_i : V_i \to W_i$, $i = 1, 2$, of vector spaces, there is a unique linear transformation $V_1 \otimes V_2 \to W_1 \otimes W_2$ which takes $x_1 \otimes x_2$ to $T_1 x_1 \otimes T_2 x_2$, for all $x_i \in V_i, i = 1, 2$.

Now suppose that the 3-dimensional complex vector space V has an endomorphism T with matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Determine the Jordan canonical form of the transformation $T \otimes T$.

(4) (a) Suppose that R is a ring and $I, J \subseteq R$ are ideals. Prove that

$$(I \cap J)(I + J) \subseteq IJ.$$

(b) Suppose that R is a principal ideal domain (PID) and $I, J \subseteq R$ are ideals. Prove that

$$(I \cap J)(I + J) = IJ.$$

(c) Suppose that R is a unique factorization domain (UFD) and let $a, b \in R$. Prove that $(a) \cap (b)$ is a principal ideal.
(d) Suppose that the UFD R is Noetherian (i.e., every ideal of R is finitely generated) and that $(I \cap J)(I + J) = IJ$ for all ideals $I, J \subseteq R$. Prove that R is a PID.

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(5) Determine the Sylvester signature of the real matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}.$$